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BASIC CONSTRUCTIONS IN THE K-THEORY OF HOMOTOPY RING SPACES

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ABSTRACT. Using the language of category theory and universal algebra we formalize the passage from the permutative category of finitely generated free R-modules to the algebraic K-theory KR of R and thus make it applicable to homotopy ring spaces. As applications we construct a Waldhausen type of algebraic K-theory for arbitrary homotopy ring spaces, show its equivalence with constructions of May and Steiner, prove its Morita invariance and show that the algebraic K-theory KX of an E_∞ ring X is itself an E_∞ ring. Finally we investigate the monomial map $Q(BX_+^*) \to KX$.

1. INTRODUCTION

It has been known for some time that the algebraic K-theory KR of a commutative ring R admits a homotopy commutative and associative product structure (e.g. see [L] or [W3]). May showed that it even has an E_∞ ring structure [M6, Proposition 10.12], but he found little reason to believe that the K-theory KX of an A_∞ or E_∞ ring X has any structure beyond that of an H-space [M6, Remark 10.3]. Fortunately, Steiner could refute this pessimistic outlook [St2]. Using a construction for algebraic K-theory different from May's he proved that for an A_∞ ring X his KX is an infinite loop space which agrees as infinite loop space with Waldhausen's algebraic K-theory $A(Y)$ of the space Y for the A_∞ ring $X = Q((\Omega Y)_+)$. Here Q is stable homotopy and $(?)_+$ the addition of an extra base point. So it is natural to ask whether KX has even more structure if X is an E_∞ ring. If one tries to attack this question using the methods of [St2] one runs into nasty bookkeeping problems making a general procedure desirable which takes care of those. Such a machine is suggested by May's passage from bipermutative categories to E_∞ spectra [M8]. The purpose of this paper is twofold. We first develop methods which adapt this passage to the theory of A_∞ and E_∞ rings. They are of interest in their own right and have been applied successfully in [FSSV]. They will also be an essential tool in the forthcoming papers [FSV, FOV, and SV6]. We then use this approach to prove

1.1 **Theorem.** *If X is an E_∞ ring so is KX.*

Throughout this paper we use A_∞ and E_∞ rings in the sense of (2.5) below which can be considered as the homotopy invariant extensions of May's

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definitions [M6, M4] based on operad pairs. Our machine produces a K-theory space KX which a priori differs from the constructions of May and Steiner, but we have

1.2 **Proposition.** *Let X be an A_∞ ring.*

(1) *KX can be realized as a plus construction. In particular, KX agrees with May's construction* [M6] *for A_∞ rings structured by operad pairs.*

(2) *KX is an infinite loop space which is equivalent as infinite loop space to Steiner's construction.*

This result also provides the first explicit proof of the equivalence of the constructions of May and Steiner. In [SV4] we showed that the space M_nX of n-squared matrices over an A_∞ ring has a natural A_∞-structure. As an immediate consequence of our machine and of (1.2) we obtain

1.3 **Proposition.** *KX is Morita invariant, i.e. there is an infinite loop equivalence $K(M_nX) \simeq KX$.*

Finally we apply our methods to the monomial map:

1.4 **Proposition.** *Let X^* the space of homotopy units in X and BX^* its classifying space. Then there is a monomial infinite loop map $Q((BX^*)_+) \to K(X)$ which is a map of E_∞ rings if X is an E_∞ ring.*

Our translation mechanism of May's techniques to the A_∞ and E_∞ world forces us to express the well-known classical constructions in abstract terms. To help the reader to keep track of what is going on we recall the steps in May's set-up, explain the idea of the translation procedure, and indicate the necessary changes (for precise definitions see §2).

(1.5) *The steps in May's construction.* 1. One starts with the permutative category $\mathscr{A}$ of finitely generated projective modules over the ring R.

2. $\mathscr{A}$ gives rise to a lax functor $A: \mathscr{F} \to \mathscr{C}at$ from the category $\mathscr{F}$ of based finite sets into the category of small categories [M7, T1].

3. One rectifies A to a genuine functor SA by Street's or Segal's rectification process [Str, Se].

4. Composition of SA with the classifying space functor gives an $\mathscr{F}$-diagram of topological spaces. Using homotopy theory of categories one checks that this $\mathscr{F}$-space is special in the sense of [MT] (a Γ-space in the notation of [Se]).

5. Each special $\mathscr{F}$-space has an associated Ω-spectrum. The 0th space of the $\mathscr{F}$-space of (4) is the algebraic K-theory KR.

We want to make these five steps accessible to A_∞ and E_∞ ring theory. The translation is as follows:

(1.6) An A_∞ ring (and similarly an E_∞ ring) is structured by a functor $X: \mathbf{\Theta} \to \mathscr{T}op$ from a theory $\mathbf{\Theta}$ which is augmented over the theory $\mathbf{\Theta}_r$ of semirings (no additive inverses are codified), and the augmentation $F_{\mathbf{\Theta}}: \mathbf{\Theta} \to \mathbf{\Theta}_r$ is a homotopy equivalence over simple morphisms in $\mathbf{\Theta}_r$.

Given a small category $\mathscr{D}$, a functor $D: \mathscr{D} \to \mathbf{\Theta}_r$, and an A_∞ ring X we can form the diagram

$$\begin{tikzcd} \mathscr{P} \arrow[r, "HD"] \arrow[d, "\nu"] \arrow[dr, phantom, "\text{pullback}"] & \Theta \arrow[r, "X"] \arrow[d, "F_\Theta"] & \mathscr{Top} \\ \mathscr{D} \arrow[r, "D"] & \Theta_r & \end{tikzcd}$$

If D takes simple morphisms as values only, ν is an equivalence so that the $\mathscr{P}$-diagram $X \circ HD$ can be considered a $\mathscr{D}$-diagram up to coherent homotopy. We can rectify it to an equivalent genuine $\mathscr{D}$-diagram using Segal's homotopy pushdown construction [Se, Appendix B] or a related process.

The simple principle is that constructions for rings which are universal, i.e. which do not depend on the particular ring and hence can be described in terms of the morphisms of Θ_r, can be performed up to coherent homotopy in the A_∞-world, expressed by the inverse images of these morphisms in Θ, and can be replaced by strict data constructions by the process (1.6), provided only simple morphisms are involved.

The $\mathscr{F}$-space associated with the algebraic K-theory of a genuine ring R has as underlying space the disjoint union of the classifying spaces $BGl_n(R)$ of the general linear groups. To allow constructions of this kind we have to enlarge Θ_r to a category $\Sigma \wr \Theta_r$ by formally adjoining all categorical sums. By taking disjoint unions in $\mathscr{Top}$ an A_∞ ring $X: \Theta \to \mathscr{Top}$ extends to a functor $\Sigma \wr \Theta \to \mathscr{Top}$, and we are in precisely the same situation as in (1.6) with Θ and Θ_r substituted by $\Sigma \wr \Theta$ and $\Sigma \wr \Theta_r$.

After the recollection in §2 of the basic definitions of A_∞ and E_∞ monoids and rings and their more combinatorial equivalent analogues, the special $\mathscr{F}$-spaces and $\mathscr{F} \wr \mathscr{F}$-spaces, we start the development of our machinery in §3 defining the substitute for $\mathscr{Cat}$, a category $\mathscr{Cat}(\Sigma \wr \Theta_r)$ of category objects in $\Sigma \wr \Theta_r$. The existence of certain iterated pullbacks in $\Sigma \wr \Theta_r$ makes this possible. The next step is to check whether Street's rectification extends to a lax functor into $\mathscr{Cat}(\Sigma \wr \Theta_r)$. This is done in §4 by translating it into category theoretical constructions and checking that those can be executed in $\mathscr{Cat}(\Sigma \wr \Theta_r)$. In Step (1.5.4) the homotopy theory in $\mathscr{Cat}(\Sigma \wr \Theta_r)$ requires some attention because Segal's pushdown construction is only natural up to homotopy with respect to subdiagrams of $\mathscr{D}$. Since we are only interested in homotopy types this suffices (see §5). The applications start with §6: We give a formal description of the permutative category of finitely generated free R-modules as an object in $\mathscr{Cat}(\Sigma \wr \Theta_r)$ (the category of finitely generated projective R-modules cannot be described in universal terms because projectivity depends on the particular ring). We then put this object through the machinery. In the comparison with Steiner's and May's constructions, §§7 and 8, we partly extend our methods: Steiner's construction, considered from our viewpoint, uses Segal's rectification of a lax functor, and the comparison of both rectifications described in [M7] can be translted into our set-up. The proofs of (1.1), ..., (1.4) are now simple consequences.

We are indebted to R. Steiner for several illuminating conversations, in particular in connection with §8. We are grateful to P. May for publishing [M8] which clarified the connection of E_∞ rings and $\mathscr{F} \wr \mathscr{F}$-spaces and described the passage from bipermutative categories to E_∞ ring spectra. Finally, we want to

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2. Homotopy monoids and rings, $\mathcal{F}$- and $\mathcal{F} \wr \mathcal{F}$-spaces

For the reader's convenience we recall the definitions of A_∞ and E_∞ monoids and rings from [SV4], where a more detailed discussion is given and the relation to other definitions in the literature is explained. The modifications needed for a module approach to the theory will be given in [SV6]. We work in the category $\mathcal{T}op$ of compactly generated topological spaces in the sense of [V].

Let Θ_m, Θ_{cm}, Θ_r, and Θ_{cr} be the theories of monoids, commutative monoids, semirings, and commutative semirings respectively. These are categories with objects $0, 1, 2, \ldots$. The morphisms from n to k in Θ_m are k-tuples of monomials

$$x_{i_1}^{r_1} \cdot \cdots \cdot x_{i_p}^{r_p} \tag{2.1}$$

in n noncommuting variables $x_1, \ldots, x_n$, in Θ_{cm} they are k-tuples of monomials

$$x_1^{r_1} \cdot \cdots \cdot x_n^{r_n} \tag{2.2}$$

in n commuting variables, in Θ_r they are k-tuples of finite sums of monomials

$$c \cdot x_{i_1}^{r_1} \cdot \cdots \cdot x_{i_p}^{r_p} \tag{2.3}$$

of type (2.1) with coefficients $c \in \mathbb{N}$, the set of nonnegative integers, and in Θ_{cr} they are k-tuples of finite sums of monomials

$$c \cdot x_1^{r_1} \cdot \cdots \cdot x_n^{r_n} \tag{2.4}$$

of type (2.2) with coefficients $c \in \mathbb{N}$. Composition is given by substitution. A morphism from n to 1 is called *simple* if all its coefficients c are ≤ 1; in the commutative cases we in addition assume that all exponents r_i are ≤ 1. A morphism from n to k is *simple* if all its k components are. In particular, every morphism in Θ_m is simple. In all other cases the simple morphisms do not form a subcategory. In the commutative monoid case there is a remedy (see (2.9)).

In the following definition Θ_* stands for $\Theta_m, \Theta_{cm}, \Theta_r, \Theta_{cr}$, whatever is appropriate, and $s\Theta_*$ its subset of simple morphisms.

2.5 **Definition.** An A_∞ or (in the commutative case) E_∞ *monoid* or *ring theory* is a topological theory Θ together with a theory functor $F = F_\Theta \colon \Theta \to \Theta_*$ such that

(1) $\operatorname{ob}\Theta \subset \operatorname{mor}\Theta$ is a closed cofibration.
(2) $F\colon \operatorname{mor}\Theta \to \operatorname{mor}\Theta_*$ is bijective on path components and a homotopy equivalence over $s\Theta_*$.

A Θ-space, i.e. a continuous functor $X\colon \Theta \to \mathcal{T}op$ such that the maps $X(n) \to (X(1))^n$ induced by the projection set operations are homotopy equivalences, is called an A_∞ or E_∞ *monoid* or *ring.*

Remark. By changing X within its homotopy type if necessary, we can arrange X to be a strictly product preserving functor [M8].

We will often find it convenient to refer to X when we mean its *underlying space* $X(1)$, and vice versa.

As maps we use h-morphisms (homotopy homomorphisms), which arise naturally when one transfers classical homomorphisms to the A_∞ or E_∞ world, or hammocks introduced in [DK], which describe the functoriality of our constructions in simple terms. For a short summary see [SV4], a more detailed study of their relationship is given in [SV5]. After restriction to a suitable universe the Θ-spaces and hammocks form a simplicial category $\mathcal{T}\!op^{\Theta}$.

A technical aspect of our constructions is the interplay of the above definition of homotopy monoids and rings with the more combinatorial descriptions of Segal [Se] and Woolfson [Wo]. We shall use the detailed description of May [M8].

Let $\mathcal{F}$ be the category of based sets $\underline{n} = \{0, 1, 2, \dots, n\}$ with basepoint 0 and based maps. We single out the morphisms

$$\pi_i = \pi_{i,n}\colon \underline{n} \to \underline{1}, \qquad 1 \le i \le n, \text{ and } \hat{\mu}_n\colon \underline{n} \to \underline{1}$$

where π_i sends i to 1 and the rest to 0 while $\hat{\mu}_n$ sends all $i > 0$ to 1.

2.6 **Definition.** An *$\mathcal{F}$-space* (Γ-space in Segal's terminology) is a functor $X\colon \mathcal{F} \to \mathcal{T}\!op$. We call $X(\underline{1})$ its *underlying space.* An $\mathcal{F}$-space is called *special,* if

(1) $(\pi_1, \dots, \pi_n)\colon X(\underline{n}) \to (X(\underline{1}))^n$ is homotopy equivalence and $X(\underline{0})$ is contractible,

(2) for each injection $\phi\colon \underline{m} \to \underline{n}$ the map $\phi\colon X(\underline{m}) \to X(\underline{n})$ is a Σ_ϕ-equivariant cofibration, where $\Sigma_\phi \subset \Sigma_n$ is the subgroup of all permutations σ satisfying $\sigma(\operatorname{Im}\phi) = \operatorname{Im}\phi$.

An $\mathcal{F}$-space, such that $(\pi_1, \dots, \pi_n)\colon X(\underline{n}) \to (X(\underline{1}))^n$ is a homeomorphism for all $\underline{n}$, is a commutative monoid with the multiplication

$$X(\underline{1})^2 \cong X(\underline{2}) \xrightarrow{\hat{\mu}_2} X(\underline{1}).$$

This is a first indication that special $\mathcal{F}$-spaces are combinatorial descriptions of E_∞ monoids. The precise passage from E_∞-monoids to $\mathcal{F}$-spaces is given by the Segal push-down [Se, Appendix B].

Segal's push-down construction.

2.7 **Definition.** A *category of operators* in the sense of [Se] is a small category $\mathcal{C}$ with topologized morphism sets and continuous composition such that $\operatorname{ob} C \subset \operatorname{mor} C$ is a closed cofibration. A *$\mathcal{C}$-space* is a continuous functor $X\colon \mathcal{C} \to \mathcal{T}\!op$. A *homomorphism* of $\mathcal{C}$-spaces is a natural transformation $\tau\colon X \to Y$ of such functors. τ is called a weak equivalence if $\tau(C)\colon X(C) \to Y(C)$ is a homotopy equivalence for all $C \in \operatorname{ob}\mathcal{C}$. An *equivalence* $F\colon \mathcal{C} \to \mathcal{D}$ *of categories of operators* is a continuous functor which is bijective on objects and a homotopy equivalence of morphism spaces.

Let $\mathcal{T}\!op^{\mathcal{C}}$ denote the category of $\mathcal{C}$-spaces. A continuous functor $F\colon \mathcal{C} \to \mathcal{D}$ induces a pull-back functor

$$F^*\colon \mathcal{T}\!op^{\mathcal{D}} \to \mathcal{T}\!op^{\mathcal{C}}.$$

Segal constructed a functor

$$F_*\colon \mathcal{T}op^{\mathscr{C}} \to \mathcal{T}op^{\mathscr{D}}$$

by defining $F_*(X) = B(\mathscr{D}, \mathscr{C}, X)$, where the right side is the two-sided bar construction. In detail, $F_*(X)(D)$ for $D \in \operatorname{ob}\mathscr{D}$ is the topological realization of the simplicial space $B_*(\mathscr{D}(-, D), \mathscr{C}, X)$ given by

$$[n] \mapsto \coprod_{C_i \in \operatorname{ob}\mathscr{C}} \mathscr{D}(F(C_n), D) \times \mathscr{C}(C_{n-1}, C_n) \times \cdots \times \mathscr{C}(C_0, C_1) \times X(C_0)$$

with the obvious simplicial structure. The $\mathscr{D}$-structure is given by composition from the left.

2.8 **Properties.** (1) F_* is a functor which preserves weak equivalences.

(2) There are homomorphisms of $\mathscr{C}$-spaces

$$F^*F_*X \xleftarrow{\alpha(X)} Id^*Id_*X \xrightarrow{\varepsilon(X)} X$$

natural in X, and $\varepsilon(X)$ is always a weak equivalence while $\alpha(X)$ is a weak equivalence if F is an equivalence.

(3) There is a natural homomorphism $\beta(Y)\colon F_*F^*Y \to Y$ for $Y \in \mathcal{T}op^{\mathscr{D}}$, which is a weak equivalence if F is an equivalence.

(4) Given a commutative diagram of categories of operators

$$\begin{array}{ccccc} \mathscr{A} & \xrightarrow{U} & \mathscr{C} & \xrightarrow{X} & \mathcal{T}op \\ \downarrow{\scriptstyle G} & & \downarrow{\scriptstyle F} & & \\ \mathscr{B} & \xrightarrow{V} & \mathscr{D} & & \end{array}$$

there is a homomorphism of $\mathscr{B}$-spaces

$$\omega(U, V)\colon G_*(X \circ U) \to F_*(X) \circ V$$

functorial in (U, V) in the obvious sense. If F and G are equivalences, $\omega(U, V)$ is a weak equivalence.

(5) If $\operatorname{mor}\mathscr{D}$ is discrete, then any monomorphism $g\colon D_1 \rightarrowtail D_2$ in $\mathscr{D}$ induces an $\operatorname{End}_g D_2$-equivariant closed cofibration

$$F_*X(g)\colon F_*X(D_1) \to F_*X(D_2)$$

where $\operatorname{End}_g D_2$ is the monoid of all $h \in \operatorname{End} D_2$ such that $F_*X(h)$ is an endomorphism of $\operatorname{Im} F_*X(g)$.

(1), (2) and (3) are proved in [Se], (4) is a consequence of (1) and (2), while (5) holds trivially on the simplicial level, because $F_*X(g)([n])$ is the inclusion of disjoint summands. Topological realization preserves the cofibration property.

Passage from E_∞ monoids to $\mathscr{F}$-spaces. We construct a functor

$$Mo\colon \mathscr{F} \to \Theta_{cm}.$$

In $\mathscr{F}$ we have a wedge sum $\underline{m} \vee \underline{n}$, which is to be identified with $\underline{m+n}$ in blocks, and a smash product $\underline{m} \wedge \underline{n}$ to be identified with $\underline{mn}$ via lexicographical ordering. A morphism $\phi\colon \underline{m} \to \underline{n}$ then decomposes into

$$\underline{m} \xrightarrow{\sigma^*} \underline{r}_0 \vee (\underline{r}_1 \vee \cdots \vee \underline{r}_u) \xrightarrow{O_{r_0} \vee id} \underline{r}_1 \vee \cdots \vee \underline{r}_u \xrightarrow{\hat{\mu}_{r_1} \vee \cdots \vee \hat{\mu}_{r_n}} \underline{l} \vee \cdots \vee \underline{l} = \underline{n}$$

where r_j is the number of elements in $\phi^{-1}(j)$ and σ^* is a suitable permutation. The functor Mo sends ϕ to $(\mu_{r_1} \times \cdots \times \mu_{r_n}) \circ \tilde{\sigma}^*$ in Θ_{cm}, where $\tilde{\sigma}^*: X^m \to X^k$; $k = \sum_{i=1}^n r_i$; is the composition of the permutation of coordinates corresponding to σ^* and the projection corresponding to O_{r_0}. The morphisms μ_{r_i} are given by $\mu_n = x_1 \cdot \cdots \cdot x_n$, for $n \geq 1$, μ_0 is the unit. It is easily seen that

(2.9) Mo maps $\mathcal{F}$ isomorphically onto a subcategory $\mathcal{S}$ of simple morphisms. In addition, $F_\Theta: \Theta \to \Theta_{cm}$ is an E_∞ monoid theory iff (2.5) holds with $s\Theta_{cm}$ replaced by $\mathcal{S}$.

Given an E_∞ monoid $X: \Theta \to \mathcal{T}op$ with $\Theta \to \Theta_{cm}$ an E_∞ monoid theory, we form the pullback

(2.10)
$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\overline{Mo}} & \Theta \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle F_\Theta} \\ \mathcal{F} & \xrightarrow{Mo} & \Theta_{cm} \end{array}$$

By (2.9), G is an equivalence, and the properties 2.8 imply that $G_*(X \circ \overline{Mo})$ is a special $\mathcal{F}$-space such that $G_*(X \circ \overline{Mo})(\underline{1}) \simeq X(1)$.

The passage from special $\mathcal{F}$-spaces to E_∞ monoids is more involved. We refer the reader to [M7] and [SV3].

The E_∞ ring case: The replacement for $\mathcal{F}$ in this case is Woolfson's category $\mathcal{F} \wr \mathcal{F}$. We use May's version of it. The category $\mathcal{F} \wr \mathcal{F}$ is a clever way of combining simple additive and simple multiplicative morphisms via the distributive law to a new category of simple morphisms. This is achieved by enlarging the set of objects in order to separate additive and multiplicative simple morphisms as much as the distributive law allows. $\operatorname{ob} \mathcal{F} \wr \mathcal{F} = \coprod_{n \geq 0} \mathbb{N}^n$, with elements denoted $(m; R)$ where $R = (\underline{r}_1, \ldots, \underline{r}_m)$. A morphism $(\phi, \varphi): (m; R) \to (n; S)$ consists of a morphism $\phi: \underline{m} \to \underline{n}$ in $\mathcal{F}$ and a collection $\varphi = (\varphi_1, \ldots, \varphi_n)$ of morphisms

$$\varphi_j: \bigwedge_{\phi(i)=j} \underline{r}_i \to \underline{s}_j$$

in $\mathcal{F}$ (the empty smash product is $\underline{1}$). Composition is the obvious one. The morphism (ϕ, φ) is called an injection and each $\varphi_j: \underline{r}_i \to \underline{s}_j$ is injective, $\phi(i) = j$; if $j \notin \operatorname{Im} \phi$, then $\varphi_j: \underline{1} \to \underline{s}_j$ can be any morphism in $\mathcal{F}$. For an injection (ϕ, φ) let $\Sigma(\phi, \varphi)$ be the group of automorphisms $(\sigma; \tau): (n; S) \to (n; S)$ such that $(\sigma; \tau) \operatorname{Im}(\phi; \varphi) = \operatorname{Im}(\phi, \varphi)$. Here we interpret (ϕ, φ) as a map of based sets

$$\varphi_1 \wedge \cdots \wedge \varphi_n: \underline{r}_{\phi^{-1}(1)} \wedge \cdots \wedge \underline{r}_{\phi^{-1}(n)} \to \underline{s}_1 \wedge \cdots \wedge \underline{s}_n$$

and similarly $(\sigma; \tau)$.

2.11 **Definition.** An *$\mathcal{F} \wr \mathcal{F}$-space* (hyper-$\Gamma$-space in Woolfson's terminology) is a functor $X: \mathcal{F} \wr \mathcal{F} \to \mathcal{T}op$. We call $X(1; \underline{1})$ its *underlying space.* An $\mathcal{F} \wr \mathcal{F}$-space is called *special,* if

(1) $X(0; *)$ contracts to a nondegenerate base point.
(2) $X(1; \underline{0}) \simeq *$.
(3) $X(1; \underline{n}) \to (X(1; \underline{1}))^n$ induced by $(id; \pi_i)$, $i = 1, \ldots, n$, is a homotopy equivalence.

(4) $X(n;S) \to \prod_{j=1}^{n} X(1;\underline{s}_j)$ induced by $(\pi_i;id)$, $i=1,\dots,n$, is a homotopy equivalence.

(5) If $(\phi;\varphi)\colon (m;R) \to (n;S)$ is an injection, then $(\phi;\varphi)\colon X(m;R) \to X(n;S)$ is a $\Sigma(\phi;\varphi)$-equivariant cofibration.

A special $\mathscr{F} \wr \mathscr{F}$-space such that the equivalences $(1),\dots,(4)$ of (2.11) are homeomorphisms determines and is determined by a topological commutative semiring [M8, 2.4]. Addition is given by $(id_1, \hat{\mu}_2)$ and multiplication by $(\hat{\mu}_2; id)$.

Passage from E_∞ rings to $\mathscr{F} \wr \mathscr{F}$-spaces. As in the E_∞ monoid case we construct a functor

$$Ri\colon \mathscr{F} \wr \mathscr{F} \to \Theta_{cr}.$$

It sends the object $(m;R)$ to $r_1+\dots+r_m$ and the morphism $(\phi,\varphi)\colon (m;R) \to (n;S)$ to the following $(s_1+\dots+s_n)$-tuple of polynomials $z_{11},\dots,z_{1s_1},\dots,z_{n1},\dots,z_{ns_n}$ in $r_1+\dots+r_m$ variables $x_{11},\dots,x_{1r_1},\dots,x_{m1},\dots,x_{mr_m}$: Recall that

$$\varphi_j\colon \bigwedge_{\phi(i)=j} \underline{r}_i \to \underline{s}_j.$$

Then

$$z_{jq} = \sum_{\varphi_j(U)=q} \prod_{\phi(i)=j} x_{iu_i}$$

where U runs over the lexicographically ordered set of sequences with ith term u_i is satisfying $1 \le u_i \le r_i$ for $i \in \phi^{-1}(j)$. Again the empty sum is 0 and the empty product is 1. Each such polynomial is simple. The equivalent of (2.9) does not hold: the image of Ri is not a subcategory and Ri does not pick up all simple morphisms in $\Theta_{cr}(m,1)$; e.g. $x_1x_2+x_1$ is not in the image.

We can proceed as in (2.10) to obtain a special $\mathscr{F} \wr \mathscr{F}$-space from an E_∞ ring. For the passage from $\mathscr{F} \wr \mathscr{F}$-spaces to E_∞ rings we use May's machinery of [M8].

2.12 *Remark.* Since the standard CW-approximation of an E_∞ theory is again an E_∞ theory, we may assume that each E_∞ structure is codified by a CW-category, and up to weak equivalence we may assume that all our E_∞ rings and $\mathscr{F} \wr \mathscr{F}$-spaces are CW-complexes.

2.13 *Remark.* The analogue of $\mathscr{F} \wr \mathscr{F}$ for A_∞ rings is $\overline{\Delta}^{\mathrm{op}} \wr \mathscr{F}$ where $\overline{\Delta}^{\mathrm{op}}$ is the image of the canonical object preserving functor $\Delta^{\mathrm{op}} \to \mathscr{F}$ [MT, 3.5]. For an analysis of $\overline{\Delta}^{\mathrm{op}}$ see [T2, p. 224]. The above definition of Ri adapts to this case and May's machinery [M8] can easily be extended to this situation.

3. Category objects in theories

In this section we introduce the framework in which we define and study the algebraic K-theory of A_∞ and E_∞ rings.

Let Θ be an A_∞ ring theory. We have shown in [SV4] that the structure of the A_∞ ring of n-squared matrices M_nX over a Θ-space X can be expressed in terms of Θ. So we have a good grip on the "linear maps" between free X-modules in terms of matrices. The idea is to give a formal description of the category of finitely generated free R-modules for a genuine ring R in terms of operations in Θ_r, which then can be lifted to Θ using the augmentation

$F_\Theta\colon \Theta \to \Theta_r$ (thus systematizing [St2]). The appropriate term is the notion of a suitable category object.

3.1 **Definition.** Let $\mathscr{C}$ be an arbitrary category. A *category object* in $\mathscr{C}$ is a simplicial object $\mathscr{A}\colon \Delta^{\mathrm{op}} \to \mathscr{C}$ in $\mathscr{C}$ such that $\mathscr{A}[n]$ is an iterated pullback of

$$\begin{tikzcd}[column sep=small]
\mathscr{A}[1] \arrow[dr, "d^1"'] & & \mathscr{A}[1] \arrow[dl, "d^0"] \arrow[dr, "d^1"'] & \cdots & \mathscr{A}[1] \arrow[dl, "d^0"] \arrow[dr, "d^1"'] & & \mathscr{A}[1] \arrow[dl, "d^0"] & (n \text{ copies}) \\
& \mathscr{A}[0] & & \cdots & & \mathscr{A}[0] & &
\end{tikzcd}$$

for $n > 1$.

3.2 *Remark.* Usually a category object in $\mathscr{C}$ consists of the object $\mathscr{A}_0$ of objects, the object $\mathscr{A}_1$ of morphisms, a source map $s = d^1\colon \mathscr{A}_1 \to \mathscr{A}_0$, a target map $t = d^0\colon \mathscr{A}_1 \to \mathscr{A}_0$, the inclusion of identities $u = s^0\colon \mathscr{A}_0 \to \mathscr{A}_1$ and a composition $c = d^1\colon \mathscr{A}_2 \to \mathscr{A}_1$ where $\mathscr{A}_2$ is a pullback of

$$\begin{tikzcd}
\mathscr{A}_1 \arrow[dr, "d^1"'] & & \mathscr{A}_1 \arrow[dl, "d^0"] \\
& \mathscr{A}_0 &
\end{tikzcd}$$

subject to the usual axioms. For the homotopy theory of categories we need its nerve, which is given in Definition 3.1, while the nerve functor in the less restricted definition is only defined up to natural isomorphism. If $\mathscr{C}$ has canonical iterated pullbacks the usual definition suffices for our purposes.

We intend to study category objects in Θ. Since we also have to codify direct sum operations, Θ is too small: it does not have categorical sums. We adopt Steiner's extension who adjoined arbitrary direct sums to Θ to obtain a category $\Sigma \wr \Theta$.

Let $\mathscr{C}$ be any category. Then $\Sigma \wr \mathscr{C}$ is the category of pairs $(S; (A_s)_{s\in S})$ consisting of a set S and an S-indexed family of objects A_s in $\mathscr{C}$. A morphism $(f, \phi)\colon (S; (A_s)) \to (T; (B_t))$ consists of a set map $f\colon S \to T$ and an S-indexed family $\phi = (\phi_s)$ of morphisms $\phi_s\colon A_s \to B_{f(s)}$ in $\mathscr{C}$. Composition is the obvious one.

$(S; (A_s))$ is the categorical sum $\coprod_{s\in S}(*; A_s)$ where $*$ is a single element set. Hence we have

3.3 **Lemma.** (1) *$\Sigma \wr \mathscr{C}$ admits arbitrary canonical sums.*

(2) *If $\mathscr{D}$ is a category with canonical sums and $F\colon \mathscr{C} \to \mathscr{D}$ is a functor, there is a canonical extension functor*

$$\coprod F\colon \Sigma \wr \mathscr{C} \to \mathscr{D}, \quad (S; (A_s)) \mapsto \coprod_{s\in S} F(A_s).$$

Now let Θ be an arbitrary theory. Let (0) stand for families consisting of copies of the object 0 of Θ. The following results are quite obvious.

3.4 **Lemma.** *The inclusion functor*

$$\mathscr{S}ets \to \Sigma \wr \Theta, \quad S \mapsto (S; (0))$$

is right adjoint to the forgetful functor

$$\Sigma \wr \Theta \to \mathcal{S}ets, \quad (S\,;\,(n_s)) \mapsto S.$$

In particular, it preserves (inverse) limits.

3.5 **Lemma.** *The inclusion functor* $\mathcal{S}ets \to \Sigma \wr \Theta$ *preserves colimits.*

3.6 **Lemma.** $(S \times T\,;\,(m_s + n_t)_{(s,t)})$ *is the product of* $(S\,;\,(m_s))$ *with* $(T\,;\,(n_t))$ *in* $\Sigma \wr \Theta$.

We also need the existence of particular pullbacks: Let $\mathcal{S}$ be the category of finite sets $\langle n \rangle = \{1, 2, \ldots, n\}$ and set maps. Let $\mathcal{S}^{\mathrm{op}} \subset \Theta$ be the subcategory of set operations $\sigma^*: m \to n$ for $\sigma: \langle n \rangle \to \langle m \rangle$ in $\mathcal{S}$. Given a diagram

$$(S, (m_s)) \xrightarrow{(f\,;\,(\varphi_s))} (T, (n_t)) \xleftarrow{(g\,;\,(\sigma_r^*))} (R, (k_r))$$

where $\sigma_r: \langle n_{g(r)} \rangle \to \langle k_r \rangle$ is an injection in $\mathcal{S}$ for all $r \in R$, we can construct a canonical pullback diagram

$$\begin{tikzcd}[column sep=large] (S \times_T R\,;\,(l_{(s,r)})) \arrow[r, "(q_2\,;\,\pi)"] \arrow[d, "(q_1\,;\,(\rho^*_{(s,r)}))"'] & (R\,;\,(k_r)) \arrow[d, "(g\,;\,(\sigma_r^*))"] \\ (S\,;\,(m_s)) \arrow[r, "(f\,;\,\varphi)"] & (T\,;\,(n_t)) \end{tikzcd} \tag{3.7}$$

where $\rho_{(s,r)}: \langle m_s \rangle \to \langle l_{(s,r)} \rangle$ again is an injection: q_1 and q_2 are the projections, $\langle l_{(s,r)} \rangle = \langle m_s \rangle \bigsqcup \langle v_r \rangle$ is the disjoint union in $\mathcal{S}$ (identified with $\langle m_s + v_r \rangle$ in blocks), where $\langle v_r \rangle$ is the "complement" of $\mathrm{Im}(\sigma_r: \langle n_{g(r)} \rangle \to \langle k_r \rangle)$, so that $\langle k_r \rangle \cong \langle n_{g(r)} \rangle \bigsqcup \langle v_r \rangle$. With these identifications $\rho_{(s,r)}: \langle m_s \rangle \to \langle l_{(s,r)} \rangle = \langle m_s \rangle \bigsqcup \langle v_r \rangle$ is the inclusion and

$$\pi_{(r,s)}: m_s \times v_r \xrightarrow{\varphi_s \times id} n_{f(s)} \times v_r \cong k_r$$

with the isomorphism induced from $\langle k_r \rangle \cong \langle n_{g(r)} \rangle \bigsqcup \langle v_r \rangle$. Here observe that $g(r) = f(s)$. Since the ρ^* again come from injections we can iterate this process to obtain

3.8 **Lemma.** $\Sigma \wr \Theta$ *has canonical iterated pullbacks of diagrams*

$$\begin{tikzcd}[column sep=small] (S_0; (n_{0,s_0})) \arrow[dr, "(f_1; \varphi_1)"'] & & (S_1; (n_{1,s_1})) \arrow[dl, "(g_1; \sigma_1^*)"] & \cdots & & (S_k; (n_{k,s_k})) \arrow[dl, "(g_k; \sigma_k^*)"] \\ & (T_1; (r_{1,t_1})) & & & (T_k; (r_{k,t_k})) \arrow[from=ul, "(f_k; \varphi_k)"'] & \end{tikzcd}$$

where the σ_i^* *are* S_i*-indexed families of set operations induced by injections.*

This result enables us to construct category objects in $\Sigma \wr \Theta$. If the source or the target map $\mathcal{A}_1 \to \mathcal{A}_0$ is of the form $(f\,;\,(\sigma^*))$ where (σ^*) is a family of set operations from injections σ, the required canonical iterated pullbacks exist. To get concise statements for our machinery, let $\mathcal{C}at(\Sigma \wr \Theta)$ denote the category of all those category objects in $\Sigma \wr \Theta$ for which the source map is of that form.

The inclusion $\mathcal{S}ets \subset \Sigma \wr \Theta$ induces an embedding $\mathcal{C}at \subset \mathcal{C}at(\Sigma \wr \Theta)$ of the category of small categories as a full subcategory. $\mathcal{C}at(\Sigma \wr \Theta)$ has products because $\Sigma \wr \Theta$ has. In particular, we can define a natural transformation $\alpha: F \to G$ of functors $F, G: \mathscr{C} \to \mathscr{D}$ in $\mathscr{C}at(\Sigma \wr \Theta)$ as an appropriate functor $\mathscr{L}_1 \times \mathscr{C} \to \mathscr{D}$, where $\mathscr{L}_n$ in the linear category

$$0 \to 1 \to 2 \to \cdots \to n.$$

(3.9)If $X: \Theta \to \mathscr{T}op$ is a Θ-space, the functor $\coprod X: \Sigma \wr \Theta \to \mathscr{T}op$ maps the canonical pullbacks (3.7) to the canonical pullbacks in $\mathscr{T}op$. Hence X induces a functor

$$\mathscr{C}at(X): \mathscr{C}at(\Sigma \wr \Theta) \to \mathscr{C}at(\mathscr{T}op)$$

into the category of topological categories. Similarly, a theory functor $F: \Theta_1 \to \Theta_2$ induces a functor

$$\mathscr{C}at(F): \mathscr{C}at(\Sigma \wr \Theta_1) \to \mathscr{C}at(\Sigma \wr \Theta_2).$$

4. Rectification of lax functors into $\mathscr{C}at(\Sigma \wr \Theta)$

In this section we extend Street's first rectification construction to arbitrary functors $A: \mathscr{C} \to \mathscr{C}at(\Sigma \wr \Theta)$. To fix notation let us recall the basic definitions.

(4.1) Let $\mathscr{C} \in \mathscr{C}at$. A *lax functor* $A: \mathscr{C} \to \mathscr{C}at(\Sigma \wr \Theta)$ is a pair of functions assigning a category object $A(n)$ to each $n \in \mathrm{ob}\,\mathscr{C}$ and a functor $A(f): A(m) \to A(n)$ to each morphism $f: m \to n$ of $\mathscr{C}$ together with natural transformations

$$\rho(n): A(id_n) \to id_{A(n)}, \quad \sigma(f, g): A(f \circ g) \to A(f) \circ A(g)$$

such that the following diagrams commute:

$$\begin{tikzcd}
A(f) \circ A(id_m) \arrow[dr, "A(f)\rho(m)"'] & A(f) \arrow[l, "\sigma(f, id_m)"'] \arrow[r, "\sigma(id_n, f)"] \arrow[d, equal] & A(id_n) \circ A(f) \arrow[dl, "\rho(n)A(f)"] \\
& A(f) &
\end{tikzcd}$$

$$\begin{tikzcd}
A(f \circ g \circ h) \arrow[r, "\sigma(f \circ g, h)"] \arrow[d, "\sigma(f, g \circ h)"'] & A(f \circ g) \circ A(h) \arrow[d, "\sigma(f, g)A(h)"] \\
A(f) \circ A(g \circ h) \arrow[r, "A(f)\sigma(g, h)"] & A(f) \circ A(g) \circ A(h)
\end{tikzcd}$$

(4.2) Let $A; B: \mathscr{C} \to \mathscr{C}at(\Sigma \wr \Theta)$ be lax functors. A (left) *lax natural transformation* $d: A \to B$ is a pair of functions assigning a functor $d(n): A(n) \to B(n)$ to each $n \in \mathrm{ob}\,\mathscr{C}$ and a natural transformation $d(f): B(f) \circ d(m) \to d(n) \circ A(f)$ to each morphism: $f: m \to n$ of $\mathscr{C}$ such that the following diagrams commute

$$\begin{tikzcd}
B(id_n) \circ d(n) \arrow[rr, "d(id_n)"] \arrow[dr, "\rho(n)d(n)"'] & & d(n) \circ A(id_n) \arrow[dl, "d(n)\rho(n)"] \\
& d(n) &
\end{tikzcd}$$

and for $g \circ f\colon m \to n \to p$

$$\begin{tikzcd}
B(g\circ f)\circ d(m) \arrow[r, "d(g\circ f)"] \arrow[dd, "\sigma(g,f)d(m)"] & d(p)\circ A(g\circ f) \arrow[dr, "d(p)\sigma(g,f)"] & \\
 & & d(p)\circ A(g)\circ A(f) \\
B(g)\circ B(f)\circ d(m) \arrow[r, "B(g)d(f)"] & B(g)\circ d(n)\circ A(f) \arrow[ur, "d(g)A(f)"'] &
\end{tikzcd}$$

One can compose lax natural transformations $d\colon A \to B$ and $e\colon B \to C$ by setting $(e\circ d)(n) = e(n)\circ d(n)$ on objects and

$$\begin{aligned}(e\circ d)(f)\colon C(f)\circ e(m)\circ d(m) &\xrightarrow{e(f)d(m)} e(n)\circ B(f)\circ d(m)\\ &\xrightarrow{e(n)d(f)} e(n)\circ d(n)\circ A(f)\end{aligned}$$

on morphisms. We obtain a category of lax functors and lax natural transformations.

(4.3) Let $d, d'\colon A \to B$ be lax natural transformations of lax functors $\mathscr{C} \to \mathscr{C}at(\Sigma \wr \Theta)$. A *natural homotopy* $\alpha\colon d \to d'$ consist of natural transformations $\alpha(n)\colon d(n) \to d'(n)$ such that

$$\begin{tikzcd}
B(f)d(m) \arrow[r, "B(f)\alpha(m)"] \arrow[d, "d(f)"'] & B(f)d'(m) \arrow[d, "d'(f)"] \\
d(n)A(f) \arrow[r, "\alpha(n)A(f)"] & d'(n)A(f)
\end{tikzcd}$$

commutes. If $\beta\colon d' \to d''$ is another such natural homotopy then $\beta\circ\alpha\colon d \to d''$ is given by $(\beta\circ\alpha)(n) = \beta(n)\alpha(n)$. If $\gamma\colon e \to e'$ is a natural homotopy of lax natural transformations $e, e'\colon B \to C$, then $\gamma\alpha\colon ed \to e'd'$ is given by

$$\begin{aligned}(\gamma\alpha)(n) &= \gamma(n)d'(n)\circ e(n)\alpha(n)\colon e(n)d(n) \to e(n)d'(n) \to e'(n)d'(n)\\ &= e'(n)\alpha(n)\circ\gamma(n)d(n)\colon e(n)d(n) \to e'(n)d(n) \to e'(n)d'(n).\end{aligned}$$

4.4 **Proposition.** *Let $\mathscr{C}$ be a small category and Θ a theory. There is a functor $A \mapsto SA$ from the category of lax functors $\mathscr{C} \to \mathscr{C}at(\Sigma\wr\Theta)$ and lax natural transformations to the category of genuine functors $\mathscr{C} \to \mathscr{C}at(\Sigma\wr\Theta)$ and genuine natural transformations with the following properties.*

(1) *For each $n \in \operatorname{ob}\mathscr{C}$ there is a pair of adjoint functors (i.e. front and back adjunction exist)*

$$\varepsilon(n)\colon SA(n) \rightleftarrows A(n)\colon \eta(n).$$

The $\eta(n)$ combine to a lax natural transformation $\eta\colon A \to SA$ such that for a lax natural transformation $d\colon A \to B$ the diagram

$$\begin{tikzcd}
A(n) \arrow[r, "d(n)"] \arrow[d, "\eta(n)"'] & B(n) \arrow[d, "\eta(n)"] \\
SA(n) \arrow[r, "Sd(n)"] & SB(n)
\end{tikzcd}$$

commutes. If A is a genuine functor then the $\varepsilon(n)$ combine to a genuine natural transformation $\varepsilon: SA \to A$.

(2) *If $F: \mathscr{D} \to \mathscr{C}$ is a functor, then $A \circ F: \mathscr{D} \to \mathscr{C}at(\Sigma \wr \Theta)$ is a lax functor and there is a natural transformation $\xi: S(A \circ F) \to S(A) \circ F$ such that*

$$
\begin{tikzcd}
 & S(A \circ F)(k) \arrow[dd, "\xi(k)"] \arrow[dr, "\varepsilon_{A\circ F}(k)"] & \\
A \circ F(k) \arrow[ur, "\eta_{A\circ F}(k)"] \arrow[dr, "\eta_A(F_k)"'] & & A \circ F(k) \\
 & S(A) \circ F(k) \arrow[ur, "\varepsilon_A(F_k)"'] &
\end{tikzcd}
$$

commutes for all $k \in \mathrm{ob}\mathscr{D}$. Moreover ξ is natural with respect to lax natural transformations $A \to B$.

(3) *Given an additional functor $G: \mathscr{E} \to \mathscr{D}$, then*

$$
\begin{tikzcd}
S(A \circ F \circ G) \arrow[rr, "\xi"] \arrow[dr, "\xi"'] & & S(A \circ F) \circ G \arrow[dl, "\xi \circ G"] \\
 & S(A) \circ F \circ G &
\end{tikzcd}
$$

commutes.

(4) *If $X: \Theta \to \mathscr{T}op$ is a Θ-space, then $\mathscr{C}at(X) \circ A: \mathscr{C} \to \mathscr{C}at(\mathscr{T}op)$ is a lax functor, and $S(\mathscr{C}at(X) \circ A) = \mathscr{C}at(X) \circ SA$. The analogous result holds for theory functors $\Theta_1 \to \Theta_2$.*

(5) *If $\alpha: d \to d'$ is a natural homotopy of lax natural transformations d, d': $A \to B$ then there is a natural homotopy $S\alpha: Sd \to Sd'$ of genuine natural transformations, i.e.* (4.3) *commutes with $Sd(f)$ and $Sd'(f)$ being identities. The correspondence $\alpha \mapsto S\alpha$ preserves both compositions of natural homotopies defined in* (4.3).

Proof. We give the constructions and leave the verification of the details to the reader. They are straightforward translations of Street's proofs once one knows the translation mechanism.

For $n \in \mathrm{ob}\mathscr{C}$ we have to define a category object $SA(n)$ in $\Sigma \wr \Theta$. We construct its nerve $SA_*(n)$:

$$SA_k(n) = \coprod_{(f_k, \dots, f_0)} P_{(f_k, \dots, f_0)}$$

taken over all composable morphisms

$$p_0 \xrightarrow{f_0} p_1 \xrightarrow{f_1} p_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{k-1}} p_k \xrightarrow{f_k} n$$

in $\mathscr{C}$ with final target n. Let s and t denote the source and target morphisms of our category objects. Then $P_{(f_k, \dots, f_0)}$ is the canonical pullback of

$$
\begin{tikzcd}[column sep=small]
A_1(p_k) \arrow[dr, "s"'] & & A_1(p_{k-1}) \arrow[dl, "A(f_{k-1})\circ t"'] \arrow[dr, "s"'] & & \cdots \arrow[dl] \arrow[dr, "s"'] & & A_1(p_1) \arrow[dl, "A(f_1)\circ t"'] \arrow[dr, "s"'] & & A_0(p_0) \arrow[dl, "A(f_0)"'] \\
 & A_0(p_k) & & A_0(p_{k-1}) & & A_0(p_2) & & A_0(p_1) &
\end{tikzcd}
$$

if $k > 0$. For $k = 0$ we have

$$\operatorname{ob} SA(n) = SA_0(n) = \coprod_{f_0 : p \to n} P_{(f_0)}$$

with $P_{(f_0)} = A_0(p)$. Source and target morphism in $SA(n)$ are induced by

$$\begin{aligned} \bar{s} &= \mathrm{proj}_1 \colon P_{(f_1, f_0)} \to P_{(f_1 \circ f_0)} = A_0(p_0), \\ \bar{t} &= t \circ \mathrm{proj}_2 \colon P_{(f_1, f_0)} \to A_1(p_1) \to A_0(p_1) = P_{(f_1)} \end{aligned}$$

where proj_1 and proj_2 are the structure maps of the pullback

$$\begin{tikzcd} P_{(f_1, f_0)} \arrow[r, "\mathrm{proj}_2"] \arrow[d, "\mathrm{proj}_1"'] & A_1(p_1) \arrow[d, "s"] \\ A_0(p_0) \arrow[r, "A_0(f_0)"] & A_0(p_1) \end{tikzcd}$$

The inclusion of the identities $i \colon SA_0(n) \to SA_1(n)$ sends $P_{(f)}$ to $P_{(f, id)}$ by the map induced by

$$A_1(p_0) \xleftarrow{\rho(p_0)} P_{(f)} = A_0(p_0) \xrightarrow{id} A_0(p_0).$$

Composition $c \colon SA_2(n) \to SA_1(n)$ decomposes into sums

$$c \colon P_{(f_2, f_1, f_0)} \to P_{(f_2, f_1 \circ f_0)}, \quad p_0 \xrightarrow{f_0} p_1 \xrightarrow{f_1} p_2 \xrightarrow{f_2} n.$$

$P_{(f_2, f_1, f_0)}$ and $A_3(p_2)$ are iterated pullbacks so that we have a commutative diagram of maps and structure maps

$$\begin{tikzcd}
P_{(f_2, f_1, f_0)} \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\text{pullback}"] & P_{(f_2 \circ f_1, f_0)} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{pullback}"] & & A_0(p_0) \arrow[d, "A(f_0)"'] \arrow[dddr, "\sigma(f_1, f_0)"] & \\
& A_1(p_1) \arrow[rr] \arrow[d, "t"] \arrow[ddr, "A(f_1)"] & & A_0(p_1) \arrow[dd, "A(f_1)"'] & \\
P_{(f_2, f_1)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & A_0(p_1) \arrow[d, "A(f_1)"] & & & \\
A_1(p_2) \arrow[r, "s"] & A_0(p_2) & A_1(p_2) \arrow[l, "t"'] \arrow[r, "s"] & A_0(p_2) & A_1(p_2) \arrow[l, "t"'] \\
& & A_3(p_2) \arrow[ull] \arrow[u] \arrow[urr] & &
\end{tikzcd}$$

Since $A_3(p_2)$ is the iterated pullback of the lower row, this diagram induces a morphism $r \colon P_{(f_2, f_1, f_0)} \to A_3(p_2)$. The solid arrow diagram

$$\begin{tikzcd}
A_3(p_2) \arrow[d, "\text{comp.}"] & P_{(f_2,f_1,f_0)} \arrow[l, "r"'] \arrow[r] \arrow[d, dashed, "c"] & P_{(f_2\circ f_1,f_0)} \arrow[d] \\
A_1(p_2) \arrow[dr, "s"'] & P_{(f_2,f_1\circ f_2)} \arrow[l, dashed] \arrow[r, dashed] \arrow[d, phantom, "\text{pullback}"] & A_0(p_0) \arrow[dl, "A(f_1\circ f_0)"] \\
 & A_0(p_2) &
\end{tikzcd}$$

commutes to induce the composition c.

This determines the nerve of $SA(n)$. For a morphism $g\colon m \to n$ in $\mathscr{C}$ the functor $SA(g)\colon SA(m) \to SA(n)$ is determined by $id\colon P_{(f_k,\dots,f_0)} \to P_{(g\circ f_k,\dots,f_0)}$.

The functor $\varepsilon(n)\colon SA(n) \to A(n)$ is evaluation given on objects by $A(f)\colon P_{(f)} = A(p_0) \to A(n)$ and on morphisms by

$$P_{(f_1,f_0)} \xrightarrow{h} A_2(n) \xrightarrow{\text{comp.}} A_1(n)$$

where h is induced by

$$\begin{tikzcd}
A_1(p_1) \arrow[d, "A(f_1)"] & P_{(f_1,f_0)} \arrow[l] \arrow[r] \arrow[d, dashed, "h"] & A_0(p_0) \arrow[d, "\sigma(f_1,f_0)"] \\
A_1(n) \arrow[dr, "s"'] & A_2(n) \arrow[l, dashed] \arrow[r, dashed] \arrow[d, phantom, "\text{pullback}"] & A_1(n) \arrow[dl, "t"] \\
 & A_0(n) &
\end{tikzcd}$$

The functor $\eta(n)\colon A(n) \to SA(n)$ is given on objects by $id\colon A_0(n) \to P_{(id)} = A_0(n)$ and on morphisms by $v\colon A_1(n) \to P_{(id,id)}$ defined by

$$\begin{tikzcd}
A_2(n) \arrow[d, "\text{comp.}"] & A_1(n) \arrow[l, "v'"'] \arrow[d, dashed, "v"] \arrow[dr, "s"] & \\
A_1(n) \arrow[dr, "s"'] & P_{(id,id)} \arrow[l, dashed] \arrow[r, dashed] \arrow[d, phantom, "\text{pullback}"] & A_0(n) \arrow[dl, "A(id)"] \\
 & A_0(n) &
\end{tikzcd}$$

where v' comes from

$$\begin{tikzcd}
& A_1(n) \arrow[dl, "id"'] \arrow[d, dashed, "v'"] \arrow[dr, "\rho(n)"] & \\
A_1(n) \arrow[dr, "s"'] & A_2(n) \arrow[l, dashed] \arrow[r, dashed] \arrow[d, phantom, "\text{pullback}"] & A_1(n) \arrow[dl, "t"] \\
& A_0(n) &
\end{tikzcd}$$

The functors $\rho(n)\colon A_0(n) \to A_1(n)$ define the back adjunction $\varepsilon(n)\circ\eta(n) \to Id$.
The front adjunction $Id \to \eta(n) \circ \varepsilon(n)$ is defined by

$$\begin{tikzcd}
A_0(n) \arrow[d, "\text{comp.}"] & P_{f_0}=A_0(p_0) \arrow[l, "A(f_0)"'] \arrow[d, dashed] \arrow[dr, "id"] & \\
A_1(n) \arrow[dr, "s"'] & P_{(id,f_0)} \arrow[l, dashed] \arrow[r, dashed] \arrow[d, phantom, "\text{pullback}"] & A_0(p_0) \arrow[dl, "A(f_0)"] \\
& A_0(n) &
\end{tikzcd}$$

where ι is the inclusion of the identities. If $d\colon A \to B$ is a lax natural transformation, then $Sd\colon SA \to SB$ is defined by the functors $Sd(n)\colon SA(n) \to SB(n)$ specified by

$$SA_1(n) = \coprod_{(f_1,f_0)} P_{(f_1,f_0)} \to SB_1(n) = \coprod_{(f_1,f_0)} P'_{(f_1,f_0)}$$

$$\begin{tikzcd}
B_2(p_1) \arrow[d, "\text{comp.}"] & P_{(f_1,f_0)} \arrow[l, "u"'] \arrow[r] \arrow[d, dashed] & A_0(p_0) \arrow[d, "d(p_0)"] \\
B_1(p_1) \arrow[dr, "s"'] & P'_{(f_1,f_0)} \arrow[l, dashed] \arrow[r, dashed] \arrow[d, phantom, "\text{pullback}"] & B_0(p_0) \arrow[dl, "B(f_0)"] \\
& B_0(p_1) &
\end{tikzcd}$$

with u defined by

$$\begin{tikzcd}
A_1(p_1) \arrow[d, "d(p_1)"] & P_{(f_1,f_0)} \arrow[l] \arrow[r] \arrow[d, dashed, "u"] & A_0(p_0) \arrow[d, "d(f_0)"] \\
B_1(p_1) \arrow[dr, "s"'] & B_2(p_1) \arrow[l, dashed] \arrow[r, dashed] \arrow[d, phantom, "\text{pullback}"] & B_1(p_1) \arrow[dl, "t"] \\
& B_0(p_1) &
\end{tikzcd}$$

If $\alpha: d \to d'$ is a natural homotopy of lax natural transformations, the natural transformations $S\alpha(n): Sd(n) \to Sd'(n)$ of the natural homotopy $S\alpha: Sd \to Sd'$ are given by

$$SA_0(n) \to SB_1(n), \quad P_{(f)} \to P_{(f,id)}$$

$f: p \to n$, given by

$$\begin{tikzcd}
B_2(p) \arrow[d, "\text{comp.}"] & P_f = A_0(p) \arrow[l, "u"'] \arrow[d, dashed] \arrow[dr, "d(p)"] & \\
B_1(p) \arrow[dr, "s"'] & P_{(f,id)} \arrow[l, dashed] \arrow[r, dashed] \arrow[d, phantom, "\text{pullback}"] & B_0(p) \arrow[dl, "B(id)"] \\
& B_0(p) &
\end{tikzcd}$$

with u from

$$\begin{tikzcd}
& A_0(p) \arrow[r, "d(p)"] \arrow[dl, "\alpha(p)"'] \arrow[d, dashed, "u"] & B_0(p) \arrow[d, "\rho(p)"] \\
B_1(p) \arrow[dr, "s"'] & B_2(p) \arrow[l, dashed] \arrow[r, dashed] \arrow[d, phantom, "\text{pullback}"] & B_1(p) \arrow[dl, "t"] \\
& B_0(p) &
\end{tikzcd}$$

Now let $F: \mathscr{D} \to \mathscr{C}$ be a functor. The natural transformation $\xi: S(A \circ F) \to SA \circ F$ is the collection of functors

$$\xi(k): S(A \circ F)(k) \to SA(Fk), \qquad k \in \operatorname{ob}\mathscr{D},$$

given on objects by $id: P_{(f)} \to P_{(Ff)}$, $f: p \to k$ in mor $\mathscr{D}$, and on morphisms by $id: P_{(f_1, f_0)} \to P_{(Ff_1, Ff_0)}$.

This specifies the construction. We should point out that all pullbacks used in this construction exist by (3.8). If $X: \Theta \to \mathscr{T}\!op$ is a Θ-space, $\coprod X$ maps these pullbacks to the corresponding canonical ones in $\mathscr{T}\!op$. Hence (4) holds. The verification of the remaining properties is a simple diagram chase.

5. Homotopy theory in $\mathrm{Cat}(\Sigma \wr \Theta_r)$

For a given A_∞ ring $X: \Theta \to \mathscr{T}\!op$ we associate to each diagram $D: \mathscr{K} \to \mathscr{C}at(\Sigma \wr \Theta_r)$ satisfying (5.2) below a diagram $[\widetilde{B}D]: \mathscr{K} \to \mathscr{T}\!op_h$ such that for any subcategory $i: \mathscr{L} \subset \mathscr{K}$ there is an isomorphism $[\widetilde{B}(D \circ i)] \to [\widetilde{B}D] \circ i$ in $\mathscr{T}\!op_h$. This correspondence is strictly functorial in X.

Let $\mathscr{A} \in \operatorname{ob}\mathscr{C}at(\Sigma \wr \Theta_r)$. The nerve of A defines a functor $\Delta^{\mathrm{op}} \to \Sigma \wr \Theta_r$. Consider the pullback

(5.1)
$$\begin{array}{ccccc} \mathscr{P}_{\mathscr{A}} & \xrightarrow{H\mathscr{A}} & \Sigma \wr \Theta & \xrightarrow{\coprod X} & \mathscr{T}\!op \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \Sigma\wr\Theta_\Theta} & & \\ \Delta^{\mathrm{op}} & \xrightarrow{\Delta\mathscr{A}} & \Sigma \wr \Theta_r & & \end{array}$$

The Segal pushdown induces a simplicial space

$$N\mathscr{A} = \nu_*\left(\coprod X \circ H\mathscr{A}\right): \Delta^{\mathrm{op}} \to \mathscr{T}\!op.$$

We are interested in the homotopy type of its realization $\widetilde{B}\mathscr{A} = |N\mathscr{A}|$. To be able to make any statements we introduce the following convention.

5.2 *Convention.* The structure maps (source, target, inclusion of identities, iterated composition) of all category objects, all functors and natural transformations studied in this section are supposed to take simple operators in $\Sigma \wr \Theta_r$ as values.

Let $\mathscr{L}_n$ denote the linear category $0 \to 1 \to \cdots \to n$. A functor $f: \mathscr{A} \to \mathscr{C}$ of category objects induces a diagram

(5.3)
$$\begin{array}{ccccc} \mathscr{P}_f & \xrightarrow{Hf} & \Sigma \wr \Theta & \xrightarrow{\coprod X} & \mathscr{T}\!op \\ \downarrow{\scriptstyle \mu} & \text{pullback} & \downarrow{\scriptstyle \Sigma\wr F} & & \\ \mathscr{L}_1 \times \Delta^{\mathrm{op}} & \xrightarrow{\Delta f} & \Sigma \wr \Theta_r & & \end{array}$$

and hence a map of simplicial spaces (for notational convenience we do not distinguish between a functor $\mathscr{L} \times \Delta^{\mathrm{op}} \to \mathscr{T}\!op$ and its adjoint $\mathscr{L} \to \mathscr{T}\!op^{\Delta^{\mathrm{op}}}$)

$$N_1 f = \mu_*\left(\coprod X \circ Hf\right): \mathscr{L}_1 \times \Delta^{\mathrm{op}} \to \mathscr{T}\!op.$$

We denote its source by $N_f\mathscr{A}$ and its target by $N_f\mathscr{C}$. By (2.8.4) we have weak equivalences $i_{\mathscr{A}}\colon N\mathscr{A} \to N_f\mathscr{A}$ and $i_{\mathscr{C}}\colon N\mathscr{C} \to N_f\mathscr{C}$. Choose any homotopy inverse of the realization of $i_{\mathscr{C}}$ to obtain a map of spaces

$$\widetilde{B}f\colon \widetilde{B}\mathscr{A} \to \widetilde{B}\mathscr{C}.$$

It depends on the choice of the homotopy inverse but its homotopy class $[\widetilde{B}f]$ is well-defined.

5.4 *Remark.* We can do better. From Segal's construction and elementary facts about simplicial spaces one deduces that $N_f\mathscr{A} = N\mathscr{A}$ and that $|i_{\mathscr{C}}|$ embeds $|N\mathscr{C}|$ as strong deformation retract into $|N_f\mathscr{C}|$. Since the space of deformation retractions is contractible, $\widetilde{B}f$ is uniquely defined up to contractible choice.

5.5 **Lemma.** (1) *For a commutative triangle of category objects and functors*

$$\begin{tikzcd}
 & \mathscr{C} \arrow[dr, "g"] & \\
\mathscr{A} \arrow[ur, "f"] \arrow[rr, "h"] & & \mathscr{D}
\end{tikzcd}$$

we have $[\widetilde{B}h] = [\widetilde{B}g] \circ [\widetilde{B}f]$.

(2) $[\widetilde{B}id_{\mathscr{A}}] = id_{\widetilde{B}\mathscr{A}}$.

Proof (1). The triangle defines a functor $T\colon \mathscr{L}_2 \to \mathscr{C}at(\Sigma\wr\Theta_r)$ inducing a triangle $NT\colon \mathscr{L}_2 \times \Delta^{\mathrm{op}} \to \mathscr{T}\!op$ of simplicial spaces. The solid arrow diagram

$$\begin{tikzcd}
 & & & N\mathscr{C} \arrow[dl, "i"'] \arrow[dr, "i"] \arrow[dddddrrr, dashed] & & & \\
 & & N_f\mathscr{C} \arrow[dr, "j"'] & & N_g\mathscr{C} \arrow[dl, "j"] \arrow[ddr, "N_1g"] & & \\
 & & & N_T\mathscr{C} \arrow[dr, "N_Tg"] & & & \\
 & N_f\mathscr{A} \arrow[r, "j"] \arrow[uur, "N_1f"] & N_T\mathscr{A} \arrow[ur, "N_Tf"] \arrow[rr, "N_Th"] & & N_T\mathscr{D} & N_g\mathscr{D} \arrow[l, "j"'] & \\
 & N_h\mathscr{A} \arrow[ur, "j"] \arrow[rrrr, "N_1h"] & & & & N_h\mathscr{D} \arrow[ul, "j"'] & \\
N\mathscr{A} \arrow[uur, "i"] \arrow[ur, "i"'] \arrow[uuuuurrr, dashed] \arrow[rrrrrr, dashed] & & & & & & N\mathscr{D} \arrow[uul, "i"'] \arrow[ul, "i"]
\end{tikzcd}$$

commutes, the various i's and j's are weak equivalences. The dotted arrows exist after realization making the corresponding squares homotopy commutative; they represent $[\widetilde{B}f]$, $[\widetilde{B}g]$, and $[\widetilde{B}h]$.

Proof (2). Let $r\colon \mathscr{L}_1 \times \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}}$ be the projection and $i_0, i_1\colon \Delta^{\mathrm{op}} \to \mathscr{L}_1 \times \Delta^{\mathrm{op}}$ the two inclusions. The diagram

$$\begin{tikzcd}
\mathscr{P}_{\mathscr{A}} \arrow[r] \arrow[d, "\nu"] \arrow[dr, phantom, "\text{pullback}"] & \mathscr{P}_{id} \arrow[r, "R"] \arrow[d, "\mu"] \arrow[dr, phantom, "\text{pullback}"] & \mathscr{P}_{\mathscr{A}} \arrow[r, "HA"] \arrow[d, "\nu"] \arrow[dr, phantom, "\text{pullback}"] & \Sigma \wr \Theta \arrow[r] \arrow[d] & \mathscr{T}\!op \\
\Delta^{\mathrm{op}} \arrow[r, "i_k"] & \mathscr{L}_1 \times \Delta^{\mathrm{op}} \arrow[r, "r"] & \Delta^{\mathrm{op}} \arrow[r, "\Delta A"] & \Sigma \wr \Theta_r &
\end{tikzcd}$$

induces a diagram of simplicial spaces

$$\begin{tikzcd}
& NA & \\
N_0A \arrow[ur, "r"] \arrow[rr, "N_1 id"] & & N_1A \arrow[ul, "r"'] \\
NA \arrow[u, "i_0"] & & NA \arrow[u, "i_1"]
\end{tikzcd}$$

where $N_1 id$ is defined as in (5.3). It is not the identity. By naturality of Segal's construction $r \circ i_k = id$. Hence r represents a homotopy inverse of $|i_k|$, and we obtain

$$[\widetilde{B}id_A] = [|r \circ N_1 id \circ i_0|] = [|r \circ i_0|] = [id]. \quad \square$$

(5.6) In general we are given a diagram $D\colon \mathscr{K} \to \mathscr{C}at(\Sigma \wr \Theta_r)$ of category objects and functors. The previous construction and the proof of (5.5) extend to produce a functor

$$[\widetilde{B}D]\colon \mathscr{K} \to \mathscr{T}\!op_h$$

which is naturally isomorphic to the composite functor

$$\mathscr{K} \xrightarrow{N_D} \mathscr{T}\!op^{\Delta^{\mathrm{op}}} \xrightarrow{\text{realiz.}} \mathscr{T}\!op \xrightarrow{\text{proj.}} \mathscr{T}\!op_h$$

where N_D is obtained from D via Segal's push-down similar to (5.1) and (5.3).

5.7 **Lemma.** *Given a product category object and its projections*

$$A \xleftarrow{\pi_1} A \times C \xrightarrow{\pi_2} C.$$

Then $([\widetilde{B}\pi_1], [\widetilde{B}\pi_2])\colon \widetilde{B}(A \times C) \to \widetilde{B}A \times \widetilde{B}C$ *is an isomorphism in* $\mathscr{T}\!op_h$.

Proof. Let $\mathscr{Q}$ denote the category $1 \leftarrow 0 \to 1'$. The projections induce a diagram

$$\begin{tikzcd}
\mathscr{P}_{\mathscr{Q}} \arrow[r, "HP"] \arrow[d, "\nu"] \arrow[dr, phantom, "\text{pullback}"] & \Sigma \wr \Theta \arrow[r, "\coprod X"] \arrow[d, "\Sigma \wr F"] & \mathscr{T}\!op \arrow[d, "p=\text{proj.}"] \\
\mathscr{Q} \times \Delta^{\mathrm{op}} \arrow[r, "\Delta P"] & \Sigma \wr \Theta_r \arrow[r, "{[\coprod X]}"] & \mathscr{T}\!op_h
\end{tikzcd}$$

Since F is bijective on path components, $\coprod X$ induces a sum and product preserving functor $[\coprod X]$. By (2.8.1) there is a natural isomorphism

$$p \circ N_P = p \circ \nu_* \left(\coprod X \circ HP\right) \cong \left[\coprod X\right] \circ \Delta P.$$

Since $[\coprod X] \circ \Delta P$ is a product diagram in $\mathcal{T}op_h$, so is $p \circ N_P$ and the result follows from (5.6). □

5.8 **Lemma.** *Let $\omega: f \to g$ be a natural transformation of functors $f, g: \mathcal{A} \to \mathcal{C}$. Then $[\tilde{B}f] = [\tilde{B}g]$.*

Proof. ω defines a functor $w: \mathcal{A} \times \mathcal{L}_1 \to \mathcal{C}$ such that

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "i_0"] \arrow[dr, "f"'] & \mathcal{A} \times \mathcal{L}_1 \arrow[d, "w"] & \mathcal{A} \arrow[l, "i_1"'] \arrow[dl, "g"] \\
& \mathcal{C} &
\end{tikzcd}$$

commutes, where i_0 and i_1 are the obvious inclusion functors. We obtain

$$[\tilde{B}f] = [\tilde{B}w] \circ [\tilde{B}i_0] \quad \text{and} \quad [\tilde{B}g] = [\tilde{B}w] \circ [\tilde{B}i_1].$$

Let $r: \mathcal{A} \times \mathcal{L}_1 \to \mathcal{A}$ and $q: \mathcal{A} \times \mathcal{L}_1 \to \mathcal{L}_1$ be the projection. Then $[\tilde{B}r]$ is left inverse to $[\tilde{B}i_k]$. Hence we are done, if $[\tilde{B}r]$ is an isomorphism. Since

$$([\tilde{B}r], [\tilde{B}q]): \tilde{B}(\mathcal{A} \times \mathcal{L}_1) \to \tilde{B}\mathcal{A} \times \tilde{B}\mathcal{L}_1$$

is an isomorphism in $\mathcal{T}op_h$, it suffices to show that $\tilde{B}\mathcal{L}_1$ is contractible. Since $\mathcal{L}_1 \in \operatorname{ob}\mathcal{C}at \subset \operatorname{ob}\mathcal{C}at(\Sigma \wr \Theta)$, the functor $\Delta\mathcal{L}_1: \Delta^{\mathrm{op}} \to \Sigma \wr \Theta_r$ factors through $\mathcal{S}ets$. In

$$\begin{tikzcd}
\Delta^{\mathrm{op}} \arrow[r, "\Delta L_1"] \arrow[d, "id"'] \arrow[dr, phantom, "\text{pullback}"] & \mathcal{S}ets \arrow[r, phantom, "\overset{j}{\subset}"] \arrow[d, "id"] \arrow[dr, phantom, "I"] & \Sigma \wr \Theta \arrow[r, "\coprod X"] \arrow[d, "\Sigma \wr F"] & \mathcal{T}op \\
\Delta^{\mathrm{op}} \arrow[r, "\Delta\mathcal{L}_1"] & \mathcal{S}ets \arrow[r, phantom, "\subset"] & \Sigma \wr \Theta_r &
\end{tikzcd}$$

the square I is a pullback because $0 \in \Theta$ is a terminal object. Hence $N\mathcal{L}_1 = id_*(\coprod X \circ j \circ \Delta\mathcal{L}_1)$, which by (2.8.2) is weakly equivalent to $\coprod X \circ j \circ \Delta\mathcal{L}_1$ which in turn is the usual nerve of $\mathcal{L}_1$. Hence $|N\mathcal{L}_1|$ is equivalent to the unit interval and contractible. □

5.9 **Corollary.** *Given a pair of adjoint functors $f: \mathcal{A} \to \mathcal{C}$ and $g: \mathcal{C} \to \mathcal{A}$, then $[\tilde{B}f]: \tilde{B}\mathcal{A} \cong \tilde{B}\mathcal{C}$ with inverse $[\tilde{B}g]$ in $\mathcal{T}op_h$.*

6. THE ALGEBRAIC K-THEORY OF A_∞ RINGS

As indicated is §3 we now give a formalized definition of the permutative category (for a definition see [M4]) of finitely generated free R-modules. We rectify the associated lax functor by Street's construction of §4. The methods of §5 then provide a functor $\mathcal{F} \to \mathcal{T}op$ which we use to define the algebraic K-theory of an A_∞ ring.

The permutative category object $\mathcal{M}$ in $\Sigma \wr \Theta_r$. The idea is to define $\mathcal{M}$ in such a way that for any genuine ring $R: \Theta_r \to \mathcal{S}ets$ the category object $\mathcal{C}at(R)(\mathcal{M}) \in \mathcal{C}at(\mathcal{S}ets) = \mathcal{C}at$ is the permutative category of finitely generated free R-modules R^n, $n = 0, 1, \ldots$, and linear maps. So we have one object R^n for each $n \in \mathbb{N}$ and the monoids M_nR of $(n \times n)$-matrices as sets of morphisms.

Consequently, $\mathscr{M}_0 = \mathrm{ob}\mathscr{M}$ is the object $(\mathbb{N}, (0))$ of $\Sigma \wr \Theta_r$ and $\mathscr{M}_1 = \mathrm{mor}\mathscr{M} = (\mathbb{N}, (n^2)_{n\in\mathbb{N}})$. Source and target morphisms

$$s, t\colon (\mathbb{N}, (n^2)) \to (\mathbb{N}, (0))$$

are given by the pairs consisting of the set map id_N and the projection operations $n^2 \to 0$ in Θ_r. In dimension p, $\mathscr{M}$ is given by $\mathscr{M}_p = (\mathbb{N}, (p \cdot n^2))$, where $p \cdot n^2$ should be considered as a p-tuple of $(n \times n)$-matrices. Hence p-fold composition

$$\mathrm{comp}\colon (\mathbb{N}, (pn^2)) \to (\mathbb{N}, (n^2))$$

consists of id_N and the sequence of p-fold matrix multiplications

$$M_n(x_1 \cdot \dots \cdot x_p)\colon pn^2 \to n^2$$

in the terminology of [SV4, §3]. The unit map

$$\overline{u}\colon \mathscr{M}_0 = (\mathbb{N}, (0)) \to \mathscr{M}_1 = (\mathbb{N}, (n^2))$$

consists of id_N and the sequence $M_n(1)$ of unit matrices.

This defines $\mathscr{M}$ as simplicial object with the boundaries induced by suitable projections and compositions, and degeneracies by $\overline{u}$. It remains to codify the permutative structure under direct sum.

$$\oplus\colon \mathscr{M}_1 \times \mathscr{M}_1 = (\mathbb{N} \times \mathbb{N}, (k^2 + l^2)_{(k,l)}) \to \mathscr{M}_1 = (\mathbb{N}, (n^2))$$

is the pair consisting of

$$f\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad (k, l) \mapsto k + l$$

and the sequence of operations $k^2+l^2 \to (k+l)^2$ given by block sum of matrices

$$(A, B) \to \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

For the matrix notation we use the following convention: In the category $\mathscr{S}$ of finite sets we identify $\langle k \rangle \times \langle l \rangle$ with $\langle k \cdot l \rangle$ by lexicographical ordering and $\langle k \rangle \sqcup \langle l \rangle$ with $\langle k + l \rangle$ in blocks. Hence we can identify an operation $k^2 + l^2 \to (k + l)^2$ with an operation

$$(\langle k \rangle \times \langle k \rangle) \sqcup (\langle l \rangle \times \langle l \rangle) \to (\langle k \rangle \sqcup \langle l \rangle) \times (\langle k \rangle \sqcup \langle l \rangle).$$

This operation has $(k + l)^2$ components, the entries of a $[(k + l) \times (k + l)]$-matrix, in $k^2 + l^2$ variables, the entries of an ordered pair of matrices (A, B) with A a $(k \times k)$-matrix and B an $(l \times l)$-matrix.

Clearly, $\oplus$ is associative. The commuting natural equivalence

$$c\colon \oplus \to \oplus \circ T,$$

where T is the switch map, is the morphism

$$\mathscr{M}_0 \times \mathscr{M}_0 = (\mathbb{N} \times \mathbb{N}, (0)) \to \mathscr{M}_1 = (\mathbb{N}, (n^2))$$

given by the set map $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $(k, l) \mapsto k + l$ and the family of operations $c_{k,l}\colon 0 \to (k + l)^2$

$$c_{k,l} = \begin{pmatrix} 0 & E_k \\ E_l & 0 \end{pmatrix}$$

where E_n stands for the n-squared unit matrix.

We choose $(\{0\}, (0)) \in \Sigma \wr \Theta_r$ as canonical terminal object. Then the unit of the permutative structure is the morphism

$$0 : (\{0\}, (0)) \to \mathscr{M}_0 = (\mathbb{N}, (0))$$

consisting of the set map $\{0\} \to \mathbb{N}$, $0 \mapsto 0$, and the operation id_0. As in the classical case we obtain

6.1 Proposition. *$\mathscr{M} = (\mathscr{M}, \oplus, 0, c)$ is a permutative category object in $\Sigma \wr \Theta_r$.*

In the commutative case, $\mathscr{M}$ can be extended to a bipermutative (for a definition see [M4]) category object $\mathscr{M}_c$ in $\Sigma \wr \Theta_{cr}$ using the tensor product of matrices:

$$\otimes : \mathscr{M}_1 \times \mathscr{M}_1 = (\mathbb{N} \times \mathbb{N})(k^2 + l^2)_{(k,l)}) \to \mathscr{M}_1 = (\mathbb{N}, (n^2))$$

consists of the set map $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $(k, l) \mapsto k \cdot l$ and the operations $k^2 + l^2 \to (k \cdot l)^2$ defined by

$$(A, B) \to A \otimes B.$$

(If we identify $(k \cdot l)^2$ with $\langle k \rangle \times \langle l \rangle \times \langle k \rangle \times \langle l \rangle$, the (i, p, j, q)th component of the tensor product is the monomial $a_{ij} \cdot b_{pq}$, where $A = (a_{ij})$ and $B = (b_{pq})$.) The tensor product is strictly associative. The commuting equivalence $c' : \otimes \to \otimes \circ T$ is the morphism

$$c' : \mathscr{M}_0 \times \mathscr{M}_0 = (\mathbb{N} \times \mathbb{N}, (0)) \to \mathscr{M}_1 = (\mathbb{N}, (n^2))$$

given by the set map $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $(k, l) \mapsto k \cdot l$ and the family of operations $c'_{k,l} : 0 \to (k \cdot l)^2$ given by the entries of the permutation matrix which maps the (i, p)th unit vector of $R^{k \cdot l}$ to the (p, i)th unit vector under the identifications

$$\langle k \rangle \times \langle l \rangle \cong \langle k \cdot l \rangle \cong \langle l \rangle \times \langle k \rangle.$$

The unit of $\otimes$ is the morphism

$$1 : (\{0\}, (0)) \to \mathscr{M}_0 = (\mathbb{N}, (0))$$

consisting of $\{0\} \to \mathbb{N}$, $0 \mapsto 1$, and the operation id_0. Again, as in the classical case, we have

6.2 Proposition. *$\mathscr{M}_c = (\mathscr{M}, \oplus, 0, c, \otimes, 1, c')$ is a bipermutative category object in $\Sigma \wr \Theta_{cr}$.*

The definition of K-theory. We proceed in accordance with the classical case (e.g. [M7, §§3, 4] or [T1]). We use the terminology of [M7]. The permutative category object $\mathscr{M}$ determines a lax functor

$$A : \mathscr{F} \to \mathscr{C}at(\Sigma \wr \Theta_r)$$

defined on objects by $A(\underline{n}) = \mathscr{M}^n$ with $\mathscr{M}^0$ the trivial category object $(\{0\}, (0))$, and on morphisms $f : \underline{m} \to \underline{n}$ by $A(f) : \mathscr{M}^m \to \mathscr{M}^n$ given by the n-tuple of morphisms

$$\mathscr{M}^m \xrightarrow{\text{proj.}} \prod_{i \in f^{-1}(j)} \mathscr{M} \xrightarrow{\oplus} \mathscr{M}, \qquad j = 1, \dots, n,$$

where the empty direct sum is the unit 0 in $\mathscr{M}$.

We apply Street's rectification of §4 to obtain a genuine functor

$$SA\colon \mathscr{F} \to \mathscr{C}at(\Sigma \wr \Theta_r).$$

Given an A_∞ ring $X\colon \Theta \to \mathscr{T}op$ with $F_\Theta\colon \Theta \to \Theta_r$ an A_∞ ring theory, we form the diagram

$$(6.3)\qquad \begin{array}{ccccc} \mathscr{P} & \xrightarrow{HSA} & \Sigma\wr\Theta & \xrightarrow{\coprod X} & \mathscr{T}op \\ \downarrow{\scriptstyle \nu} & \text{pullback} & \downarrow{\scriptstyle \Sigma\wr F} & & \downarrow{\scriptstyle \pi_0} \\ \mathscr{F}\times\Delta^{\mathrm{op}} & \xrightarrow{\Delta SA} & \Sigma\wr\Theta_r & \xrightarrow{\coprod \pi_0 X} & \mathscr{T}op_h \end{array}$$

where $\pi_0 X$ is the space of path components of X with the quotient topology (of course, $\pi_0 X$ is discrete if X is of the homotopy type of a CW-complex). Since F is bijective on path components, $\coprod \pi_0 X$ is defined by $\coprod X$. A check through the definition of $\mathscr{M}$, the rectification process and the nerve construction shows that ΔSA takes simple morphisms as values. Hence ν is an equivalence of categories. Hence Segal pushdown defines a functor

$$N_{SA}(X) = \nu_*\left(\coprod X \circ HSA\right)\colon \mathscr{F}\times\Delta^{\mathrm{op}} \to \mathscr{T}op$$

together with weak equivalences of $\mathscr{P}$-spaces

$$N_{SA}(X)\circ\nu \leftarrow id_*\left(\coprod X\circ HSA\right) \to \coprod X\circ HSA.$$

6.4 **Lemma.** *The topological realization $|N_{SA}(X)|\colon \mathscr{F} \to \mathscr{T}op$ of $N_{SA}(X)$ is a special $\mathscr{F}$-space.*

Proof. For simplicity we drop (X) from the notation. We have to show that the maps $\pi_i\colon N_{SA}(\underline{n}) \to N_{SA}(\underline{1})$ of (2.6) induce a homotopy equivalence $N_{SA}(\underline{n}) \to (N_{SA}(\underline{1}))^n$ after realization. Let Q be the composite

$$Q\colon \mathscr{F} \xrightarrow{N_{SA}} \mathscr{T}op^{\Delta^{\mathrm{op}}} \xrightarrow{\text{realiz.}} \mathscr{T}op \xrightarrow{\text{proj.}} \mathscr{T}op_h.$$

By (5.6), Q is naturally equivalent to the functor $[\widetilde{B}SA]\colon \mathscr{F} \to \mathscr{T}op_h$. If $\eta\colon A \to SA$ is the lax natural transformation of (4.4.1) then for each $\pi = \pi_i$ the diagram

$$\begin{array}{ccc} A(\underline{n}) & \xrightarrow{\eta(\underline{n})} & SA(\underline{n}) \\ \downarrow{\scriptstyle A(\pi)} & & \downarrow{\scriptstyle SA(\pi)} \\ A(\underline{1}) & \xrightarrow{\eta(\underline{1})} & SA(\underline{1}) \end{array}$$

commutes up to the natural transformation $\eta(\pi)$. In $\mathscr{T}op_h$ we hence obtain a commutative diagram

$$\begin{array}{ccccc} \widetilde{B}A(\underline{n}) & \xrightarrow[\cong]{[\widetilde{B}\eta(\underline{n})]} & \widetilde{B}SA(\underline{n}) & \cong & Q(\underline{n}) \\ \downarrow{\scriptstyle [\widetilde{B}A(\pi)]} & & \downarrow{\scriptstyle [\widetilde{B}SA(\pi)]} & & \downarrow{\scriptstyle Q\pi} \\ \widetilde{B}A(\underline{1}) & \xrightarrow[\cong]{[\widetilde{B}\eta(\underline{1})]} & \widetilde{B}SA(\underline{1}) & \cong & Q(\underline{1}) \end{array}$$

Since the $\eta(\underline{n})$ have adjoints, $[\widetilde{B}\eta(\underline{n})]$ is an isomorphism. Combining these diagrams we get a commutative diagram in $\mathcal{T}op_h$

$$\begin{array}{ccc} \widetilde{B}A(\underline{n}) & \cong & Q(\underline{n}) \\ {\scriptstyle ([\widetilde{B}A\pi_1],\dots,[\widetilde{B}A\pi_n])}\downarrow & & \downarrow{\scriptstyle (Q\pi_1,\dots,Q\pi_n)} \\ \widetilde{B}A(\underline{1})^n & \cong & Q(\underline{1})^n \end{array}$$

Since $A(\underline{n}) = A(\underline{1})^n$ and the π_i are the projections, the left arrow is an isomorphism by (5.7).

For the same reasons $[\widetilde{B}\eta(\underline{0})]: \widetilde{B}A(\underline{0}) \cong Q(\underline{0})$. Since $\Delta A(\underline{0})$ is the constant simplicial object on $(\{0\}, (0))$ in $\Sigma \wr \Theta_r$ and since X is product preserving, $\coprod X \circ HA(\underline{0})$ is the constant functor on a point. Hence $\widetilde{B}A(\underline{0})$ is contractible by (2.8). □

$|N_{SA}(X)|$ is the A_∞ analogue of the $\mathcal{F}$-space associated with the permutative category of finitely generated free R-modules R^n and *all* linear maps. For K-theory we have to restrict to linear isomorphisms, i.e. invertible matrices in our model. The A_∞ analogue of the general linear group Gl_nR is the space of homotopy invertible matrices over the A_∞ ring X, denoted by $\widetilde{Gl}_nX$ [W1]. It is the pullback

$$\begin{array}{ccc} \widetilde{Gl}_nX & \longrightarrow & M_nX = X(n^2) \\ \downarrow & & \downarrow \\ Gl_n(\pi_0X) & \subset & \pi_0(M_nX) = M_n(\pi_0X) \end{array}$$

where M_nX is considered as space of all $(n \times n)$-matrices over X. Since π_0X is a genuine semiring, $Gl_n(\pi_0X)$ is the classical general linear group.

For the passage from all matrices to homotopy invertible ones we have to analyze the composite functor $\pi_0 \circ N_{SA}(X): \mathcal{F} \times \Delta^{\mathrm{op}} \to \mathcal{S}ets$. Since

$$\pi_0 \circ N_{SA}(X) \circ \nu \cong \pi_0 \circ \coprod X \circ HSA = \coprod \pi_0 X \circ \Delta SA \circ \nu \tag{6.5}$$

and since ν is an equivalence, it suffices to consider

$$\coprod \pi_0X \circ \Delta SA: \mathcal{F} \times \Delta^{\mathrm{op}} \to \Sigma \wr \Theta_r \to \mathcal{S}ets,$$

which is the nerve of the functor (4.4.4)

$$\mathcal{C}at(\pi_0X) \circ SA = S(\mathcal{C}at(\pi_0X) \circ A): \mathcal{F} \to \mathcal{C}at(\Sigma \wr \Theta_r) \to \mathcal{C}at.$$

Denote the composite lax functor $\mathcal{C}at(\pi_0X) \circ A: \mathcal{F} \to \mathcal{C}at$ by A_0, for short. By construction, A_0 is the lax functor associated with the permutative category with $\mathbb{N}$ as set of objects and $\coprod M_n(\pi_0X)$ as set of morphisms. Composition is given by matrix multiplication. Let GA_0 be the lax subfunctor of A_0 associated with the permutative subcategory defined by $\coprod Gl_n(\pi_0X)$. The rectification $Sj: SGA_0 \to SA_0$ of the inclusion transformation $GA_0 \to A_0$ defines inclusion functors $Sj(\underline{n}): SGA_0(\underline{n}) \to SA_0(\underline{n})$. Consequently, $\Delta SGA_0: \mathcal{F} \times \Delta^{\mathrm{op}} \to \mathcal{S}ets$ is a subfunctor of ΔSA_0. We restrict $N_{SA}(X)$ to the subfunctor $G(X)$ corresonding to ΔSGA_0 under the isomorphism (6.5).

Restriction to the appropriate subfunctors in the proof of (6.4) yields

6.6 **Lemma.** *$|G(X)|: \mathscr{F} \to \mathscr{T}\!op$ is a special $\mathscr{F}$-space.*

6.7 **Definition.** The *algebraic K-theory of the A_∞ ring $X: \Theta \to \mathscr{T}\!op$* is the 0th space of the Ω-spectrum $E|G(X)|$ associated with the special $\mathscr{F}$-space $|G(X)|$. (In particular KX is an infinite loop space.)

All our constructions are natural with respect to homomorphisms of Θ-spaces and preserve weak equivalences of Θ-spaces. Hence

6.8 **Proposition.** *The correspondence $X \mapsto KX$ is functorial with respect to homomorphisms of Θ-spaces and with respect to hammocks of Θ-spaces.*

6.9 *Remark.* A homotopy homomorphism $f: X \to Y$ in the sense of [SV4; Definition 2.5] induces a map $Kf: KX \to KY$ (see [SV2] for its construction and the functorial behaviour of K with respect to homotopy homomorphisms). If $\mathscr{U}$ is the universal A_∞ ring theory derived from Steiner's canonical operad pair [SV1, §5], the A_∞ structures of X and Y pullback uniquely up to homotopy through A_∞ rings, i.e. a homotopy through continuous product preserving functors, to $\mathscr{U}$-structures making f a homotopy homomorphism of $\mathscr{U}$-spaces. This f "decomposes" canonically into a hammock

$$X \xleftarrow{r(X)} UX \xrightarrow{Uf} Y,$$

i.e. $r(X)$ and Uf are homomorphisms of $\mathscr{U}$-spaces and $r(X)$ is an equivalence. Hence our construction defines a map

$$KX \simeq K(UX) \to KY$$

which is uniquely determined by f up to homotopy and homotopy equivalence of source and target (see (6.10) below). This passage to hammocks makes the study of homotopy homomorphisms and their unpleasant functorial properties redundant.

In view of this remark the following result is worth noticing for coherence investigations.

6.10 **Proposition.** *Let $X_t: \Theta \to \mathscr{T}\!op$, $t \in [0, 1]$, be a homotopy through A_∞ rings. Then there is an infinite loop space $K(X_I)$ and infinite loop inclusions $i_t: K(X_t) \to K(X_I)$ as strong deformation retracts. The i_t and the retractions are natural with respect to homotopies of homomorphisms $X_t \to Y_t$ of Θ-spaces. In particular, $K(X_0) \simeq K(X_1)$ as infinite loop spaces.*

Proof. Define X_I by $X_I(n) = X(n) \times [0, 1]$.

7. The equivalence with Steiner's definition

While our approach to KX for an A_∞ ring X is in the spirit of [T1 and M7], based on Street's rectification of lax functors, Steiner's construction [St2] follows Segal's passage from permutative categories to special $\mathscr{F}$-spaces [Se]. Segal's method can be viewed as an alternative rectification of the lax functor $A: \mathscr{F} \to \mathscr{C}\!at$ defined by the given permutative category. The formalization of this rectification in [M5] cannot be generalized to our situation because it explicitly uses the isomorphisms in the categories $A(\underline{m})$ which are not accessible in the abstract context. In the special case of the permutative category object $\mathscr{M}$ of §6 one can get around this problem by using the isomorphisms defined by permutation matrices. This is the idea behind Steiner's construction of a functor $\mathscr{F} \times \Delta^{\mathrm{op}} \to \Sigma \wr \Theta_r$.

We start with the definition of the functor

$$C = StA\colon \mathscr{F} \to \mathscr{C}at(\Sigma \wr \Theta_r)$$

implicit in Steiner's set-up, where A is the lax functor of §6.

Recall, that $\underline{r}$ denotes the based set $\{0, 1, \ldots, r\}$. Let $V(m)$ be the set of all families $(r_1, \ldots, r_m; (\pi_T))$ where $r_i \in \mathbb{N}$ and (π_T) is a family of bijections

$$\pi_T\colon \underline{r}_{t_1} \vee \cdots \vee \underline{r}_{t_p} \to \underline{r_{t_1} + \cdots + r_{t_p}} \tag{7.1}$$

indexed by all subsets $T = \{0, t_1, \ldots, t_p\} \subset \underline{m}$, where $\underline{0}$ stands for the empty wedge. We assume that $\pi_T = id$ if $T = \{0, t\}$. Let $E(m)$ be the set of all families $(r_1, \ldots, r_m; (\pi_T), (\pi'_T))$ where (π_T) and (π'_T) are both families of bijections of type (7.1). The category object $C(\underline{m})$ is defined by

$$C_0(\underline{m}) = (V(m); (0)), \quad C_1(\underline{m}) = (E(m); (r_1^2 + \cdots + r_m^2)_{(r_1, \ldots, r_m; (\pi), (\pi'))}).$$

The source and target maps are determined by

$$E(m) \to V(m); \quad (r_1, \ldots, r_m; (\pi), (\pi')) \mapsto (r_1, \ldots, r_m; (\pi)),$$
$$\text{resp. } (r_1, \ldots, r_m; (\pi')).$$

Composition is defined by $(r_1, \ldots, r_m; (\pi), (\pi'), (\pi''))$ and matrix multiplication $M_{r_i} \times M_{r_i} \to M_{r_i}$.

To define the $\mathscr{F}$-structure let $\phi\colon \underline{m} \to \underline{n}$ be in $\mathscr{F}$. We associate a map

$$V(m) \to V(n), \quad (r_1, \ldots, r_m, (\pi_r)) \mapsto (s_1, \ldots, s_n; (\rho_U))$$

with $s_i = \sum_{\phi(j)=i} r_j$ and for $U = \{0, \overline{u}_1, \ldots, \overline{u}_p\} \subset \underline{n}$

$$\rho_U\colon \bigvee_{i \in U^*} \underline{s}_i \xrightarrow{\bigvee \pi_{s_i}^{-1}} \bigvee_{i \in U^*} \bigvee_{\phi(j)=i} \underline{r}_j = \bigvee_{\phi(j) \in U} \underline{r}_j \xrightarrow{\pi_{\phi^{-1}(U)}} \underline{s_{\overline{u}_1} + \cdots + s_{\overline{u}_p}} \tag{7.2}$$

with $U^* = U \backslash \{0\}$. Analogously we define a map $E(m) \to E(n)$.

The functor $C(\phi)\colon C(\underline{m}) \to C(\underline{n})$ is determined by $C_1(\underline{m}) \to C_1(\underline{n})$, given by the map $E(m) \to E(n)$ and the following operations indexed by $(r_1, \ldots, r_m; (\pi), (\pi'))$

$$r_1^2 + \cdots + r_m^2 \to s_1^2 + \cdots + s_n^2.$$

Its ith component sends the m-tuple of matrices $(A_1, \ldots, A_m)$ to the $s_i \times s_i$-matrix associated to the linear map of free R-modules

$$R^{s_i} \xrightarrow{P(\pi_{\phi^{-1}(i)})^{-1}} \bigoplus_{\phi(j)=i} R^{r_j} \xrightarrow{\oplus A_j} \bigoplus R^{r_j} \xrightarrow{P(\pi'_{\phi^{-1}(i)})} R^{s_i} \tag{7.3}$$

where $P(\pi)$ are the permutation matrices determined by π (here we have to identify $\bigvee_{\phi(j)=i} \underline{r}_j$ with an object of $\mathscr{F}$ in our canonical way to obtain an ordered base of $\bigoplus R^{r_j}$).

Following May [M7, Appendix] we construct lax natural transformations $\delta\colon C \to A$ and $\nu\colon A \to C$. The functor $\delta(\underline{m})\colon C_1(\underline{m}) \to A_1(\underline{m}) = M_1^m$ is given by

$$E(m) \to \mathbb{N}^m, \quad (r_1, \ldots, r_m; (\pi), (\pi')) \mapsto (r_1, \ldots, r_m)$$

and the operations $id\colon r_1^2+\cdots+r_m^2 \to r_1^2+\cdots+r_m^2$. For $\phi\colon \underline{m}\to\underline{n}$ the natural transformation $\delta(\phi)\colon A(\phi)\circ\delta(\underline{m}) \to \delta(\underline{n})\circ C(\phi)$ is determined by composition with permutation matrices according to (7.3).

The functor $\nu(\underline{m})\colon A_1(\underline{m}) \to C_1(\underline{m})$ sends $(r_1, \dots, r_m)$ to $(r_1, \dots, r_m; (\kappa), (\kappa))$ where (κ) is the family of canonical identification. (Here, as in (7.3) we have to order the wedge which we do by the canonical ordering of the indices.) The natural transformation $\nu(\phi)\colon C(\phi)\circ\nu(\underline{m}) \to \nu(\underline{n})\circ A(\phi)$ is given by composition with the permutation matrix determined by the reordering of the double wedge in (7.2).

(7.4) $\delta\circ\nu = Id$, and for each $\underline{m}$ there is a natural isomorphism $\xi(\underline{m})\colon Id \to \nu(\underline{m})\circ\delta(\underline{m})$. The $\xi(\underline{m})$ combine to a natural homotopy $\xi\colon Id\to\nu\circ\delta$.

The $\xi(\underline{m})$ are induced by the maps $V(m)\to V(m)$, $(r_1, \dots, r_m; (\pi)) \mapsto (r_1, \dots, r_m; (\kappa))$ and the associated permutation matrices $P(\pi)$. The statements of (7.4) are easily checked.

Steiner proceeds with StA in the same way as we did with SA in §6 to define a K-functor $K_{St}X$. Consider the diagram

$$\begin{array}{ccccc} SA(\underline{m}) & \xrightarrow{S\nu(\underline{m})} & S(StA)(\underline{m}) & \xrightarrow{\varepsilon(\underline{m})} & StA(\underline{m}) \\ {\scriptstyle \eta_A(\underline{m})}\uparrow & & {\scriptstyle \eta_{StA}(\underline{m})}\uparrow & & \\ A(\underline{m}) & \xrightarrow{\nu(\underline{m})} & StA(\underline{m}) & & \end{array}$$

By (4.4.1) the square commutes and, in the notation of §5.65, the maps $\widetilde{B}\nu(\underline{m})$, $\widetilde{B}\eta_A(\underline{m})$, $\widetilde{B}\eta_{StA}(\underline{m})$, and $\widetilde{B}\varepsilon(\underline{m})$ are homotopy equivalences. We obtain a natural transformation

$$\varepsilon\circ S\nu\colon SA\to StA$$

which induces a weak equivalence of the associated $\mathscr{F}$-spaces

$$|N_{SA}(X)| \to |N_{StA}(X)|.$$

Restriction to "invertible components" preserves this equivalence, and we obtain

7.5 **Proposition.** *There is an infinite loop equivalence $KX \to K_{St}X$, natural with respect to homomorphisms and hammocks of Θ-spaces.*

8. KX AS PLUS CONSTRUCTION

An important result of the algebraic K-theory KR of a ring R is that it can be expressed as the plus construction on $BGlR = \operatorname{colim} BGl_nR$. It "reduces" the calculation of the homology of KR to the one of $BGlR$. For strictly product preserving A_∞ rings X May has given a plus construction version K_MX of algebraic K-theory [M6]. We will show that KX is equivalent to K_MX, which establishes this result for A_∞ rings.

Steiner in [St2, Remark 3.5] and in private conversations expected that a comparison of the telescope implicit in his construction with the telescope used by May in [M6] will yield the proof but did not provide the details for technical reasons. The problem is to define a reasonable map $KX \to K_MX$ (which then almost automatically is a homotopy equivalence). We start with an explicit description of the telescope defind by a special $\mathscr{F}$-space Y following [Se, §4].

$Y(\underline{1})$ has a homotopy commutative and associative H-space structure defined by

$$\mu\colon Y(\underline{1}) \times Y(\underline{1}) \overset{(\pi_1,\pi_2)}{\underset{\simeq}{\longleftarrow}} Y(\underline{2}) \xrightarrow{\hat{\mu}_2} Y(\underline{1}).$$

In our cases $\pi_0 Y(\underline{1}) \cong \mathbb{N}$ as abelian monoid, and we denote the path component corresponding to $n \in \mathbb{N}$ by Z_n. Choose $z \in Z_1$ and define $i_n\colon Z_n \to Z_{n+1}$ by

$$\begin{tikzcd} i_n : Z_n \arrow[r, "{(id, z)}"] & Z_n \times Z_1 \arrow[r, dashed] \arrow[d, phantom, "\cap"] & Z_{n+1} \arrow[d, phantom, "\cap"] \\ & Y(\underline{1}) \times Y(\underline{1}) \arrow[r, "\mu"] & Y(\underline{1}) \end{tikzcd}$$

8.1 **Lemma.** *Let $\mathbb{B}Y$ denote the infinite loop space associated with Y. Then*

$$\mathbb{Z} \times (\operatorname{Tel} Z_n)^+ \simeq \mathbb{B}Y$$

May's proof [M2, §15] applies to our situation although μ is not strictly associative.

Although $\operatorname{Tel} Z_n$ depends on the particular choice of $z \in Z_1$ and of the homotopy inverse of (π_1, π_2) in the definition of μ, its homotopy type is uniquely determined by Y.

In the case of $Y = |N_{SA}(X)|$ recall the diagram

$$(8.2)\qquad \begin{tikzcd} SA(\underline{1}) \times SA(1) & SA(\underline{2}) \arrow[l, "{(SA\pi_1, SA\pi_2)}"'] \arrow[r, "SA(\hat{\mu}_2)"] & SA(\underline{1}) \\ A(\underline{1}) \times A(\underline{1}) \arrow[u, "\eta(\underline{1})\times\eta(\underline{1})"'] \arrow[r, equal, "{(A\pi_1, A\pi_2)=id}"] & A(\underline{2}) \arrow[u, "\eta(\underline{2})"'] \arrow[r, "A\hat{\mu}_2"] & A(\underline{1}) \arrow[u, "\eta(\underline{1})"'] \end{tikzcd}$$

which after application of the functor $[\widetilde{B}]$ of §5 becomes a commutative diagram in $\mathcal{T}op_h$. The top row induces the multiplication μ on $\widetilde{B}SA(\underline{1})$. Hence the telescope associated with $|N_{SA}(X)|$ is equivalent to the telescope obtained from the lower row of (8.2): By (5.7) we have a morphism

$$\widetilde{B}A(\underline{1}) \times \widetilde{B}A(\underline{1}) \cong \widetilde{B}A(\underline{2}) \xrightarrow{[\widetilde{B}\oplus]} \widetilde{B}A(\underline{1})$$

in $\mathcal{T}op_h$. By [M1, 11.11]

$$\pi_0 \widetilde{B}\mathcal{C} = \pi_0 |N\mathcal{C}| = \pi_0 |\pi_0 N\mathcal{C}|, \qquad \mathcal{C} \in \mathcal{C}at(\Sigma \wr \Theta_r).$$

Hence $\pi_0[\widetilde{B}\bigoplus]\colon \pi_0\widetilde{B}A(\underline{1}) \times \pi_0\widetilde{B}A(\underline{1}) \to \pi_0\widetilde{B}A(\underline{1})$ is equivalent to π_0 applied to the realization of

$$\mathcal{L}_1 \times \Delta^{\mathrm{op}} \xrightarrow{F} \Sigma \wr \Theta_r \xrightarrow{\coprod \pi_0 X} \mathcal{T}op$$

by (2.8.2). Here F is defined by the functor $A(\hat{\mu}_2)$ in (8.2). By construction, this is

$$\left(\coprod_{n\in N} BM_n(\pi_0 X)\right) \times \left(\coprod_n BM_n(\pi_0 X)\right) \xrightarrow{B\oplus} \coprod_n BM_n(\pi_0 X).$$

Hence the telescope can be described as follows: Define $T\colon \mathbb{N} \times \Delta^{\mathrm{op}} \to \Theta_r$ by $T(k, [n]) = n \cdot k^2$ considered as an n-tuple of $k \times k$-matrices $(A_1, \dots, A_n)$.

The boundary maps d^i are given by matrix multiplication $A_i \cdot A_{i+1}$ for $0 < i < n$ and by the projections deleting A_1 resp. A_n for $i = 0$ and n. The degeneracies insert unit matrices. For $k < l$, the operation $n \cdot k^2 \to n \cdot l^2$ substitutes each matrix A_i by

$$\begin{pmatrix} A_i & 0 \\ 0 & E_{l-k} \end{pmatrix}.$$

(8.3) The telescope obtained from $\mathbb{N} \xrightarrow{N_T} \mathcal{T}\!op^{\Delta^{\mathrm{op}}} \xrightarrow{\text{realiz.}} \mathcal{T}\!op$ where N_T comes from the Segal pushdown of

$$\begin{tikzcd} \wp T \arrow[r, "T'"] \arrow[d, "\nu"] \arrow[dr, phantom, "\text{pullback}"] & \Theta \arrow[r, "X"] \arrow[d, "F"] & \mathcal{T}\!op \\ \mathbb{N} \times \Delta^{\mathrm{op}} \arrow[r, "T"] & \Theta_r & \end{tikzcd}$$

is equivalent to the telescope associated with SA (i.e. the telescope of $|N_{SA}(X)|$).

For the construction of $K_M X$ we use the reformulation in [SV4, §4]. Let Θ_m be the theory of monoids. There is a canonical functor $i\colon \Delta^{\mathrm{op}} \to \Theta_m$ sending $[n]$ to n, boundaries $d^i\colon [n] \to [n-1]$ to the operations $(x_1, \dots, x_i \cdot x_{i+1}, \dots, x_n)$ for $0 < i < n$, and to the projections deleting x_1 resp. x_n if $i = 0$ and n etc. The functor T factorizes as

$$\mathbb{N} \times \Delta^{\mathrm{op}} \xrightarrow{id \times i} \mathbb{N} \times \Theta_m \xrightarrow{T_m} \Theta_r$$

where T_M is the functor constructed in [SV4, 4.1]. We obtain a diagram

$$\begin{tikzcd} \wp T \arrow[r, "J"] \arrow[d, "\nu"] \arrow[dr, phantom, "\text{pullback}"] & A_*\Theta \arrow[r, "T'_M"] \arrow[d, "\rho"] \arrow[dr, phantom, "\text{pullback}"] & \Theta \arrow[r, "X"] \arrow[d, "F"] & \mathcal{T}\!op \\ \mathbb{N} \times \Delta^{\mathrm{op}} \arrow[r, "id \times i"] & \mathbb{N} \times \Theta_m \arrow[r, "T_M"'] & \Theta_r & \end{tikzcd}$$

and $A_*\Theta$ is an $\mathbb{N}$-coloured A_∞ monoid theory. If $W(\mathbb{N} \times \Theta_m)$ is the universal $\mathbb{N}$-coloured A_∞ monoid theory of [BV], there is a theory functor H: $W(\mathbb{N} \times \Theta_m) \to A_*\Theta$, unique up to homotopy through functors, such that $\rho \circ H = \varepsilon\colon W(\mathbb{N} \times \Theta_m) \to \Theta_m$, the universal augmentation [BV, 3.20]. In [SV4] we apply the rectification process M of [BV, 4.49] to obtain an $\mathbb{N}$-indexed sequence of monoids and homomorphisms. Restriction to the invertible components corresponding to the groups $Gl_n(\pi_0 X)$ gives a sequence

$$R_0 X \to R_1 X \to R_2 X \to \cdots$$

of monoids and cofibration homomorphisms such that $R_n X$ contains the A_∞ monoid $\widetilde{Gl}_n X$ as strong deformation retract. Then

$$K_M X = K_0(\pi_0 X) \times (B\widetilde{Gl}X)^+,$$

where $B\widetilde{Gl}X = \operatorname{colim} BR_n$. So we have to compare the nerve of the M-construction $M(X \circ T'_M \circ H)$, which is $M(X \circ T'_M \circ H) \circ (id \times i)$, with $\nu_*(X \circ T')$. Consider

$$\begin{tikzcd}
\wp_1 \arrow[rrr, "Q"] \arrow[dr, "\xi"] \arrow[dd, dashed, "\lambda"] & & & W(\mathbb{N}\times\Theta_m) \arrow[dl, "H"', "\simeq"] \arrow[dd, dashed, "\varepsilon", "\simeq"'] \\
& \wp_T \arrow[r, "J"] \arrow[dl, "\nu"] \arrow[ur, phantom, "\text{pullback}"] \arrow[dr, phantom, "\text{pullback}"] & A_*\Theta \arrow[dr, "\rho", "\simeq"'] & \\
\mathbb{N}\times\Delta^{\mathrm{op}} \arrow[rrr, "id\times i"'] & & & \mathbb{N}\times\Theta_m
\end{tikzcd}$$

By (2.8.4) we have weak equivalences of $\mathbb{N}\times\Delta^{\mathrm{op}}$-spaces

$$\nu_*(X\circ T')\to\rho_*(X\circ T'_M)\circ(id\times i)\leftarrow(\rho\circ H)_*(X\circ T'_M\circ H)\circ(id\times i).$$

Hence it suffices to compare $(\rho\circ H)_*(X\circ T'_M\circ H)=\varepsilon_*(X\circ T'_M\circ H)$ with the M-construction. So let $Y=X\circ T'_M\circ H\colon W(\mathbb{N}\times\mathbf{\Theta}_m)\to\mathcal{T}\!op$. The M-construction comes equipped with a universal homotopy homomorphsm $Y\to\varepsilon^*MY$. By [BV, 4.23] and (2.8.2) we have a sequence of $W(\mathbb{N}\times\mathbf{\Theta}_m)$-homomorphisms

$$\varepsilon^*\varepsilon_*Y\leftarrow id^*id_*Y\to Y\leftarrow UY\to\varepsilon^*MY$$

which are weak equivalences. Apply the Segal pushdown ε_* to transform them into weak $\mathbb{N}\times\Delta^{\mathrm{op}}$-equivalences. Since there are weak equivalences

$$\varepsilon_*Y\leftarrow\varepsilon_*\varepsilon^*\varepsilon_*Y,\quad \varepsilon_*\varepsilon^*MY\to MY$$

by (2.8.3), we are done.

8.4 **Proposition.** *There is a chain of homotopy equivalences* $KX\simeq\mathbb{Z}\times(B\widetilde{Gl}X)^+$, *natural with respect to homomorphisms of* $\mathbf{\Theta}$*-spaces. In particular,*

$$\pi_iKX\cong\pi_iK_MX\quad \textit{for } i>0.$$

9. Morita invariance

In [SV4] we showed that the space M_kX of $k\times k$-matrices over a strictly product preserving A_∞ ring X has a canonical A_∞ ring structure. This proof literally goes through with the present more general definition of an A_∞ ring. In this section we prove

9.1 *Morita invariance.* There is a homotopy equivalence of infinite loop spaces $K(M_kX)\simeq KX$, natural with respect to homomorphisms of $\mathbf{\Theta}$-spaces and hammocks of $\mathbf{\Theta}$-spaces.

We construct a permutative category object $\mathcal{M}at$ in $\Sigma\wr\mathbf{\Theta}_r$, which corresponds to the permutative category of free M_kR-modules $(M_kR)^n$ for a genuine ring R, by putting

$$\mathcal{M}at_0=(\mathbb{N},(0)),\quad \mathcal{M}at_1=(\mathbb{N},(kn)^2_{n\in N}).$$

We have to think of $\mathcal{M}at_1$ as the union over $n\in\mathbb{N}$ of all $n\times n$-matrices with $k\times k$-matrices as entires. Since a matrix of matrices is a matrix, there is an inclusion $J\colon\mathcal{M}at\to\mathcal{M}$. The permutative category structure on $\mathcal{M}at$ is induced from the one on $\mathcal{M}$, so that J is a permutative functor.

If $M_n\colon \Theta_r \to \Theta_r$ is the product preserving functor of [SV4, §3], then $\mathscr{M}at$ is the image of $\mathscr{M}$ under $\Sigma \wr M_n$, and J is induced by the inclusion of the image. From [SV4] we obtain a diagram

$$\begin{tikzcd} \wp \arrow[r, "N'"] \arrow[d, "\nu"'] \arrow[dr, phantom, "\text{pullback}"] & \Sigma\wr M_n\Theta \arrow[r, "\Sigma\wr\widehat{M}_n"] \arrow[d, "\Sigma\wr F_{M_n}"] \arrow[dr, phantom, "\text{pullback}"] & \Sigma\wr\Theta \arrow[r, "\coprod X"] \arrow[d, "\Sigma\wr F"] & \mathscr{T}op \\ \mathscr{F}\times\Delta^{\mathrm{op}} \arrow[r, "\mathrm{Nerve}\circ SA"] & \Sigma\wr\Theta_r \arrow[r, "\Sigma\wr M_n"] & \Sigma\wr\Theta_r & \end{tikzcd} \tag{9.2}$$

and Street's rectification of the lax functor A_{mat} arising from $\mathscr{M}at$ composed with the nerve is nothing but the lower row of (9.2). Since $K(M_kX)$ is defined from the left square of (9.2) and since J is a functor of permutative category objects, we obtain a natural transformation of functors $SA_{mat} \to SA\colon \mathscr{F} \to \mathscr{C}at(\Sigma\wr\Theta_r)$, which induces an infinite loop map $K(M_kX) \to KX$. The induced map of associated telescopes maps the nth space $\widetilde{Gl}_n(M_kX)$ to the $(k\cdot n)$th space $\widetilde{Gl}_{n\cdot k}(X)$. Hence, for cofinality reasons, the map of associated telescopes is a homotopy equivalence. The result now follows from (8.4).

10. The E_∞ ring case

We make extensive use of [M8]. Throughout this section let $X\colon \Theta \to \mathscr{T}op$ be an E_∞ ring.

The bipermutative category object $\mathscr{M}_c$ of (6.2) gives rise to a lax functor (compare [M8, §3])

$$P\colon \mathscr{F}\wr\mathscr{F} \to \mathscr{C}at(\Sigma\wr\Theta_{cr})$$

such that

$$A\colon \mathscr{F} \xrightarrow{J} \mathscr{F}\wr\mathscr{F} \xrightarrow{P} \mathscr{C}at(\Sigma\wr\Theta_{cr})$$

is the functor described in §6. Here J is the inclusion functor sending $\underline{n}$ to $(\underline{1};(\underline{n}))$ and $\varphi\colon \underline{m} \to \underline{n}$ to $(id_{\underline{1}};(\varphi))$. We apply Street's rectification and obtain a functor

$$\mathscr{F}\wr\mathscr{F}\times\Delta^{\mathrm{op}} \xrightarrow{\Delta SP} \Sigma\wr\Theta_{cr}.$$

As in §6 it takes simple morphisms as values. Segal's pushdown provides us with a functor (depending on X)

$$N_{SP} = N_{SP}(X)\colon \mathscr{F}\wr\mathscr{F} \to \mathscr{T}op^{\Delta^{\mathrm{op}}}.$$

Using the results of §5 we obtain

(10.1) *The realization $|N_{SP}(X)|$ is a special $\mathscr{F}\wr\mathscr{F}$-space.* We restrict $|N_{SP}(X)|$ to invertible components by restricting the monoids $M_n(\pi_0X)$ to the monoids $Gl_n(\pi_0X)$ as in §6 and obtain a special $\mathscr{F}\wr\mathscr{F}$-space

$$|G_c(X)|\colon \mathscr{F}\wr\mathscr{F} \to \mathscr{T}op.$$

By [M8, §4], the spectrum $E(|G_c(X)|\circ J)$ is an E_∞ ring spectrum and its 0th space $E_0(|G_c(X)|\circ J)$ an E_∞ ring whose structure is codified by Steiner's canonical operad pair [St1]. Since $A = P\circ J$, there is a weak equivalence of special $\mathscr{F}$-spaces $|G(X)| \to |G_c(X)|\circ J$ by (2.8.4), and hence an infinite loop equivalence.

$$KX \to E_0(|G_c(X)|\circ J). \tag{10.2}$$

By construction, the infinite loop structure on KX is codified as an additive E_∞ monoid by the additive operad of Steiner's operad pair. By homotopy invariance of E_∞ ring structures [SV1, 4.8] we can extend the infinite loop structure on KX to an E_∞ ring structure making (10.2) an E_∞ ring equivalence. In particular,

10.3 **Theorem.** *If X is an E_∞ ring, KX is an E_∞ ring, too.*

11. The monomial map

Let $X\colon \Theta \to \mathcal{T}op$ be an A_∞ ring and $X^* = \widetilde{Gl}_1 X \subset X(1)$ its homotopy units. To simplify the argument we assume that X is a strictly product preserving A_∞ ring. Then X^* is an A_∞ monoid under multiplication. In [M6] May constructed a map $Q((BX^*)_+) \to KX$, where $Q = \Omega^\infty S^\infty$ and $Z_+ = Z \cup \{\text{point}\}$, called *monomial map* because it is induced by the inclusion of the homotopy invertible monomial $n \times n$-matrices into $\widetilde{Gl}_n X$, but as he pointed out in [M8, Appendix B.6.5] there was a flaw in his construction. The monomial map is of importance because it relates stable homotopy to K-theory and it occurs in the construction of the Waldhausen splitting [W2]

$$A(Y) \simeq Q(Y_+) \times \mathrm{Wh}\, Y.$$

In [SV4] we constructed the monomial map following the suggestions of May, but it can more easily be described in our present context.

A matrix is called *monomial* if it has at most one nonzero entry in each row and column.

Let $\mathcal{M}_1$ be the trivial category object in $\Sigma \wr \Theta_r$ defined by $\mathrm{ob}\mathcal{M}_1 = (\{1\}, 0)$ and $\mathrm{mor}\mathcal{M}_1 = (\{1\}, 1)$. The inclusion $\{1\} \subset \mathbb{N}$ and the operation $id_1 \in \Theta_r$ define an inclusion functor $J\colon \mathcal{M}_1 \to \mathcal{M}$ into the category object $\mathcal{M}$ of §6. If $\mathcal{E} \subset \mathcal{F}$ denotes the subcategory of all permutations we can form the category object $\mathcal{E} \wr \mathcal{M}_1$ as in [M8, Remark 3.5]. It is permutative in the A_∞ case and bipermutative in the E_∞ case and J extends to a strict permutative respectively bipermutative functor

$$J\colon \mathcal{E} \wr \mathcal{M}_1 \to \mathcal{M}.$$

We treat the E_∞ ring case. The arguments in the A_∞ ring case are analogous and considerably simpler. J induces a strict transformation of lax functors $j\colon U \to \grave{A}\colon \mathcal{F} \wr \mathcal{F} \to \mathcal{C}at(\Sigma \wr \Theta_{cr})$ and Street's rectification then gives a natural transformation $S_j\colon SU \to SA$. Since all data consist of simple morphisms, Segal pushdown yields a diagram of $(\mathcal{F} \wr \mathcal{F} \times \Delta^{\mathrm{op}})$-spaces

$$N_{SU} \xrightarrow{\alpha} N_{Sj}SU \xrightarrow{N_{Sj}} N_{Sj}SA \xleftarrow{\beta} N_{SA} \tag{11.1}$$

where N_{Sj} is induced by Sj and α, β are the weak equivalences given by (2.8.4).

The telescope implicit in $|N_{SU}|$ is equivalent to the one constructed from

$$U(\underline{1};\underline{1}) \times U(\underline{1};\underline{1}) = U(\underline{1};\underline{2}) \xrightarrow{(id\,;\,\hat{\mu}_2)} U(\underline{1};\underline{1}).$$

If R is a genuine ring, $\mathrm{mor}\, U(\underline{1};\underline{1})$ is mapped to $\coprod \Sigma_n \wr R$ under $\coprod R\colon \Sigma \wr \Theta_{cr} \to \mathcal{T}op$. Here Σ_n denotes the symmetric group and $\Sigma_n \wr R$ the wreath product of Σ_n and R with

$$(\sigma; r_1, \dots, r_n) \cdot (\tau; r'_1, \dots, r'_n) = (\sigma \circ \tau; r_{\tau(1)} \cdot r'_1, \dots, r_{\tau(n)} \cdot r'_n). \tag{11.2}$$

If we identify $(\sigma; r_1, \dots, r_n)$ with the monomial $(n \times n)$-matrix having r_k as $(\sigma(k), k)$th entry, then (11.2) is matrix multiplication. Hence the telescope is equivalent to the stabilization telescope $B(\Sigma_\infty \wr X)$ of the $B(\Sigma_n \wr X)$ (compare §8), where $\Sigma_n \wr X$ is the A_∞ monoid of monomial matrices over X constructed in [SV4].

Restriction to invertible matrices and topological realization transforms (11.1) into a diagram of special $\mathscr{F} \wr \mathscr{F}$-spaces

$$|GSU(X)| \xrightarrow{\alpha} |G_jSU(X)| \xrightarrow{G_j} |G_j(X)| \xleftarrow{\beta} |G_c(X)|.$$

Passage to the 0th space of the associated E_∞ ring spectra defines a map of E_∞ rings

$$E_0|GSU(X)| \to KX$$

uniquely up to homotopy (depending on choices of a homotopy inverse of β and the choice of the homotopy inverse of $|G(X)| \to |G_c(X)| \circ J$ in §8). Moreover,

$$E_0|GSU(X)| \simeq \mathbb{Z} \times B(\Sigma_\infty \wr X^*)^+, \tag{11.3}$$

where $B(\Sigma_\infty \wr X^*)$ is the stabilization telescope of the classifying spaces $B(\Sigma_n \wr X^*)$ of homotopy invertible monomial matrices, i.e. monomial matrices with nonzero entries in X^*.

In the A_∞ ring case, these results were established by Steiner in his framework [St2, Theorem 3.6].

By an extension of the Barratt-Priddy-Quillen-Theorem there is a homotopy equivalence $\mathbb{Z} \times B(\Sigma_\infty \wr X^*)^+ \simeq Q(BX^*_+)$ [M6, Theorem 8.3], so that $E_0|GSU(X)| \simeq Q(BX^*_+)$. This result can be improved: $Q(BX^*_+)$ has a canonical E_∞ ring structure (resp. infinite loop structure in the A_∞ ring case), and we want to show that $E_0|GSU(X)| \simeq Q(BX^*_+)$ as E_∞ ring (resp. infinite loop space). Let us recall the structure of $Q(BX^*_+)$. Consider

$$\begin{tikzcd}
\wp \arrow[r, "HMo"] \arrow[d, "\nu"'] \arrow[dr, phantom, "\text{pullback}"] & A\Theta \arrow[r, "Hm"] \arrow[d, "F_m"'] \arrow[dr, phantom, "\text{pullback}"] & \Theta \arrow[r, "X"] \arrow[d, "F"] & \mathcal{T}op \arrow[d, "\pi_0"] \\
\mathscr{F} \arrow[r, "Mo"] & \Theta_{cm} \arrow[r, "m"'] & \Theta_{cr} \arrow[r, "\pi_0 X"'] & \mathcal{S}ets
\end{tikzcd}$$

where m is the inclusion of the multiplicative commutative monoid structure. Then $A\Theta$ is an E_∞ monoid theory. Restriction of $X \circ Hm$ to the components of the units of $\pi_0 X$ defines X^* as a grouplike E_∞ monoid. There are various equivalent notions of BX^* with an E_∞ monoid structure:

Define $Y: \mathscr{F} \to \mathcal{T}op$ to be the topological realization of

$$\mathscr{F} \times \Delta^{\mathrm{op}} \xrightarrow{id \times i} \mathscr{F} \times \mathscr{F} \xrightarrow{\text{smash}} \mathscr{F} \xrightarrow{\nu_*(X^* \circ HMo)} \mathcal{T}op.$$

Let $(\mathscr{H}, \mathscr{L})$ denote Steiner's canonical operad pair [St1]. The category of operators $\widehat{\mathscr{L}}$ associated to $\mathscr{L}$ augments over $\mathscr{F}$ so that Y is an $\widehat{\mathscr{L}}$-space. Y gives rise to an $\widehat{\mathscr{L}} \wr \Pi$-space in the sense of [M8], and application of [M8, 4.3] transforms Y into an $(\Pi, \mathscr{L})$-space VY. We define BX^* to be the $\mathscr{L}$-space

$$BX^*: \mathscr{L} \subset \mathscr{L} \wr \Pi \xrightarrow{R''VY} \mathcal{T}op$$

in the notation of [M8, §4]. By [M8, 4.3]; [MT], and [SV3], BX^* is homotopy equivalent to the classifying spaces of X^* in the sense of [M1] and [BV].

Ring multiplication defines both composition and tensor product in the category object $\mathscr{M}_1$. Hence the functor

$$\mathscr{F} \times \Delta^{\mathrm{op}} \xrightarrow{id \times i} \mathscr{F} \times \mathscr{F} \xrightarrow{\text{smash}} \mathscr{F} \xrightarrow{m \circ M \circ} \Theta_{cr} \subset \Sigma \wr \Theta_{cr}$$

coincides with the nerve of

$$\mathscr{F} \xrightarrow{\mu} \mathscr{F} \wr \mathscr{F} \xrightarrow{U} \mathscr{C}at(\Sigma \wr \Theta_{cr})$$

where μ is determined by $\underline{n} \mapsto (\underline{n}; (1, \ldots, 1))$. Since $U \circ \mu$ is a genuine functor, naturality of Street's construction provides natural transformations

$$U \circ \mu \overset{\varepsilon(U \circ \mu)}{\longleftarrow} S(U \circ \mu) \xrightarrow{\xi} SU \circ \mu$$

inducing weak equivalences of $(\mathscr{F} \times \Delta^{\mathrm{op}})$-spaces

$$N_{U \circ \mu} \leftarrow N_{S(U \circ \mu)} \rightarrow N_{SU \circ \mu} \rightarrow N_{SU} \circ (\mu \times id_{\Delta^{\mathrm{op}}}).$$

Restriction to invertible components and topological realization transforms this sequence into a sequence of weak equivalences of $\mathscr{F}$-spaces

$$Y \leftarrow |GU\mu(X)| \leftarrow |GS(U \circ \mu)(X)| \rightarrow |G(SU \circ \mu)(X)| \rightarrow |GSU(X)| \circ \mu$$

which puts us into the situation of the proof of [M8, 8.6]. Following the proof we obtain maps of $\mathscr{L}$-spectra

$$\Sigma^\infty(BX^*_+) \overset{\simeq}{\leftarrow} \Sigma^\infty V|GS(U \circ \mu)(X)|(\underline{1}) \rightarrow E|GSU(X)|.$$

The composite map is an equivalence on the 0th space by (11.3). Hence we have shown

11.4 **Proposition.** *There is an infinite loop equivalence $E_0|GSU(X)| \simeq Q(BX^*_+)$, which is an equivalence of E_∞ rings if X is an E_∞ ring.*

11.5 **Corollary.** *The monomial map induces an infinite loop map $Q(BX^*_+) \rightarrow K(X)$, which is a map of E_∞ rings if X is an E_∞ ring.*

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